Hedging Options with Scale-Invariant Models

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**Optimal Hedging and Scale Invariance:**
**A Taxonomy of Option Pricing Models**

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Abstract

A price process is scale-invariant if and only if the returns distribution is independent of the price level. We show that scale invariance preserves the homogeneity of a pay-off function throughout the life of the claim and hence prove that standard price hedge ratios for a wide class of contingent claims are model-free. Since options on traded assets are normally priced using some form of scale-invariant process, e.g. a stochastic volatility, jump diffusion or Lévy process, this result has important implications for the hedging literature. However, standard price hedge ratios are not always the optimal hedge ratios to use in a delta or delta-gamma hedge strategy; in fact we recommend the use of minimum variance hedge ratios for scale-invariant models. Our theoretical results are supported by an empirical study that compares the hedging performance of various smile-consistent scale-invariant and non-scale-invariant models. We find no significant difference between the minimum variance hedges in the smile-consistent models but a significant improvement upon the standard, model-free hedge ratios.

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1. Introduction

This paper extends the work of Merton (1973) on scale-invariant processes and the result of Bates (2005) on the model-free property of delta and gamma under such processes. Merton showed that standard European and American option values are homogeneous of degree one in the underlying price and strike when the option is priced under a scale-invariant process. Bates proved that if an option price is homogeneous of degree one then its delta and gamma are model-free in the class of scale-invariant processes. That is, every scale-invariant process gives the same delta and gamma for the option and the only difference between these theoretical hedge ratios and the empirically observed deltas and gammas is due to a different quality of the models’ fit to market data.

The pursuit of efficient hedge ratios is a genuine concern of any options trading desk. For instance, by hedging options dynamically, it is possible to remove the influence from the underlying price itself and place a direct bet on volatility or correlation. With the increasing liquidity of vanilla options a multitude of ‘smile-consistent’ option pricing models have emerged in the literature. Stochastic volatility, stochastic interest rates and jumps are concepts now familiar to any serious options trader and the literature on pricing models based on these is overwhelming. By contrast the literature on their hedging properties is scarce and contradictory. This paper provides the following contributions to the options hedging literature: We prove that scale-invariant models, a class wide enough to include most options pricing models in current use, have identical price hedging properties for any claim with a homogeneous pay-off function. We show that, within the class of scale-invariant models, the minimum variance price hedge ratios are different from standard price hedge ratios whenever there is a non-zero correlation between the price and another stochastic component in the model, such as in the stochastic volatility model of Heston (1993), among others. Finally we investigate, both theoretically and empirically, the optimality of standard and minimum variance hedge ratios.

The first part of this paper provides a general definition of a scale-invariant process and investigates its properties. This definition admits a wide class of stochastic processes as scale-invariant. A key property is that scale invariance preserves homogeneity, in the sense that if a pay-off function is homogeneous of degree \( k \) in the variables in the price dimension, then the price of a European or American option at any time prior to expiry is
also homogeneous of degree $k$. This property is used to extend Bates’ result on standard options to a model-free rule for the hedge ratios of path-dependent options, such as barriers, Asians, look-backs and forward starts, and of options with pay-offs that are homogeneous of degree $k \neq 1$, such as binary options and power options.

We then build on the work of Schweizer (1991) and Frey (1997) to obtain explicit expressions for the minimum variance delta and gamma of some standard option pricing models, scale-invariant or otherwise. If local minimization of variance is the paramount reason for dynamic hedging, the minimum variance delta is the hedge ratio that reduces the instantaneous covariance between the underlying asset and the hedged portfolio value to zero, and similarly for the gamma. We show that these hedge ratios require a simple adjustment to the model-free delta and gamma of scale-invariant models to account for extra dynamic features, such as price-volatility correlation.

Next, an empirical study on standard European options on the S&P 500 index indicates that extending the definition of delta and gamma from simple partial derivatives to the minimum variance hedge ratios mentioned above yields a major improvement in hedging performance. However, we find no significant difference between the performances of different minimum variance hedges. Finally, our results are not conclusive about the superiority of minimum variance hedge ratios over the Black-Scholes deltas and gammas. In fact, we find that the Black-Scholes model performs remarkably well for delta-gamma hedging of standard European options.

The rest of this paper is structured as follows. Section 2 introduces a general definition of scale-invariant processes, proves that both implied and local volatilities will be invariant under scaling and classifies some popular option pricing models as scale-invariant or otherwise. Section 3 extends Bates’ result on the model-free delta and gamma for scale-invariant processes, as explained above. We also discuss our results on the scale invariance of implied and local volatilities, and in particular their implications for hedging. Section 4 derives expressions for the minimum variance delta and gamma of some option pricing models, both scale-invariant and non-scale-invariant. Section 5 presents the results of the empirical study of hedging European options on the S&P 500 index and section 6 concludes.
2. Scale-invariant Processes

Merton (1973) identified ‘constant returns to scale’ as a desirable feature for pricing options. If the probability density of the underlying asset returns is invariant under scaling then the price of a standard American or European option will scale with the underlying price. Put another way, it does not matter whether the asset price is measured in dollars or in cents – the relative value of an option (as a percentage of the underlying price) should remain the same. Hoogland and Neumann (2001) explore the economic intuition for scale invariance, and rely on this to derive an alternative to martingale theory as a pricing tool. Whilst these authors consider scale invariance as a parallel to a change of numeraire, we regard scale invariance as the invariance of the returns density under a change in the unit of measurement of the underlying price. This is not the same as a change of numeraire. The price of every asset in the economy changes if we change the numeraire, whilst here scale invariance refers only to a change in the unit for measuring the underlying price and everything else that is in the same dimension as this price, such as an option strike or barrier. A simple example is a stock split. After the split, the value of the stock and the strike price of any option on this stock will be scaled, but the prices of the remaining assets in the economy (e.g. bonds) are not changed.

Which price processes are scale-invariant? Geometric Brownian motions, even with mean reversion or a diffusion displacement, are. The long-term average in the mean reversion mechanism is normally measured in the price dimension, as is the constant in a displaced diffusion (for instance, it may represent the firm’s level of debt when pricing an equity option). Hence tradable assets such as currencies, equities, equity indexes and commodities would normally be modelled using scale-invariant processes. But economic variables such as interest rates, volatility and inflation will not necessarily be modelled using a scale-invariant process, as these need not be driven by geometric Brownian motion. The following general definition of a scale-invariant process allows one to classify a pricing model as scale-invariant or otherwise:\footnote{Remark on Notation: The time dependence of variables is not made explicit when it is clear from the context. Hence, a variable such as $S$ stands for $(S_t)_{t \geq 0}$ unless otherwise stated.}

**Definition.** A price process $S = (S_t)_{t \geq 0}$ is scale-invariant if and only if it can be written in the form:

$$\frac{dS}{S} = G'(\omega) \, dt \quad t \geq 0$$

(1)
where $\mathbf{G}$ is a vector of time-varying dimensionless coefficients and $\mathbf{\omega} = (\omega_t)_{t \geq 0}$ is a vector of factors driving the asset price containing the time $t$, Wiener processes and/or jump processes. The coefficients in $\mathbf{G}$ may be random or deterministic, but (1) must satisfy the conditions for $S$ to be a semi-martingale.

Therefore, a price process is scale-invariant if it is a proper semi-martingale and if relative price increments are dimensionless in the unit of measurement of $S$.\(^2\) Note that it does not even need to be Markovian. Examples of option pricing models based on scale-invariant processes include Merton’s (1976) jump-diffusion and most stochastic volatility models, even with stochastic interest rates, as has already been observed by Bates (2005). Our definition allows further models to be classified as scale-invariant, including: local volatility models in which the volatility depends only on the relative price $S/S_0$, mixture diffusions (such as in Brigo and Mercurio, 2002), uncertain volatility models (Avellaneda et al., 1995), volatility jump models (Naik, 1993), and Lévy processes (Schoutens, 2003) if the drift and Lévy density are dimensionless. Option pricing models that are not scale-invariant include the constant elasticity of variance model of Cox (1975), the ‘stochastic-αβρ’ (SABR) model of Hagan et al. (2002) and the local volatility models of Dupire (1994) and Derman and Kani (1994). We now use this definition to derive important results on the volatility of scale-invariant processes:

**Proposition 1.** The following properties are equivalent:

(i) $S$ is generated by a scale-invariant process.

(ii) $0(K, T; t, S) = 0(nK, T; t, nS) \quad n \in \mathbb{R}^+$

(iii) $\hat{\sigma}(K, T; t, S) = \hat{\sigma}(nK, T; t, nS) \quad n \in \mathbb{R}^+$

where $0(K, T; t, S)$ is the implied volatility of a standard European option with strike $K$ and maturity $T$, $\hat{\sigma}(K, T; t, S)$ is the local volatility for an asset level $K$ at future time $T$, and both volatilities are calculated at time $t$ ($0 \leq t \leq T$).

**Proof.**

(i) $\iff$ (ii): The implied volatility is the volatility parameter in the Black and Scholes (1973) model that equates the Black-Scholes price $f^{hs}$ to the price $f$ of a standard European option

\(^2\) We say that a variable or function is dimensionless if it is invariant after scaling the unit of measurement of all variables in the same dimension as the price $S$. Some examples are $1_{\{S_t > S_0\}}$, $\log(S/K)$ and $\log(S/S_0)$. 

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That is:
\[ f(t, S; K, T) = f^h_t \left( t, S; K, T, \theta(K, T; t, S) \right). \] (2)

Intuitively, when the process volatility is time varying, either stochastic or deterministic, the implied volatility is a single number that represents the average volatility of the underlying asset price over the life of a European option. When the price process is scale-invariant, Merton (1973) proves that the price of a standard European option is homogeneous of degree one, so we may write:
\[ f(t, S; K, T) = S f_t \left( t, 1; \frac{K}{S}, T \right) \]
and, because the Black-Scholes price is also homogeneous of degree one:
\[ f^h_t (t, S; K, T, \theta(K, T; t, S)) = S f^h_t \left( t, 1; \frac{K}{S}, T, \theta(K, T; t, S) \right) \]
Substituting the above into (2) yields:
\[ f \left( t, 1; \frac{K}{S}, T \right) = f^h_t \left( t, 1; \frac{K}{S}, T, \theta(K, T; t, S) \right). \]

Hence \( \theta(K, T; t, S) \) is implicitly defined in terms of \( K/S \), hence it is homogeneous of degree zero in \( S \) and \( K \). Conversely, if the implied volatility is homogeneous of degree zero, then (2) implies that the option price \( f \) will be homogeneous of degree one because the Black-Scholes price is homogeneous of degree one in \( S \) and \( K \). Thus, by Proposition 2 in Section 3 below, the process must be scale-invariant.

(i) \( \Leftrightarrow \) (iii): The local volatility function may be specified explicitly, as in Dumas et al. (1998), Brigo and Mercurio (2002) and many others, or otherwise in terms of the conditional expectation of the instantaneous variance of log-returns (Dupire, 1996; Derman and Kani, 1998). See also Skiadopoulos (2001) and Fengler (2005, ch. 3) for a review. It follows from a theorem of Gyöngy (1986) that, for every Itô process on the relative price with stochastic volatility, there is a deterministic process with the local volatility defined above that admits the same marginal relative price density for every future time \( T \). Now it follows from our definition that the local volatility must be dimensionless if this deterministic process is to be scale-invariant, and vice-versa.

The proposition shows that if one of the properties (i) – (iii) holds, then so do the other two. The scale-invariant volatility properties (ii) and (iii) will be used for our analysis of the hedging properties of scale-invariant models in the next section. But first we discuss their
general implications. Property (ii) implies that the implied volatility is a function of moneyness only, as measured by $K/S$. Thus the partial derivative of the implied volatility with respect to $S$ is given by Euler’s theorem as:

$$\theta_s\left(K,T;\tau,S\right) = -\left(\frac{K}{S}\right)\theta_K\left(K,T;\tau,S\right)$$

and since $\theta_K$ is the slope of the implied volatility smile in the strike metric, the implied volatility sensitivity to $S$ is ‘model-free’ if the smile is observable in the market.

Property (iii) implies that the local volatility surface of a scale-invariant model moves as the underlying asset price changes. For instance, consider a scale-invariant local volatility model and note that at some calibration time $t$ when the underlying asset price is at $S$, the at-the-money local volatility is $\hat{\sigma}\left(S,t;S_t,S\right) = \hat{\sigma}\left(1,t;1,1\right)$, by property (iii). The local volatility for any future time $T > t$ and price $uS$ is $\hat{\sigma}\left(uS,T;T_s,S\right) = \hat{\sigma}\left(u,T;1,1\right)$. Now suppose we move in time and recalibrate the model at time $T$ when the asset price is indeed at $S_T = uS$. Again using property (iii) the at-the-money local volatility must be $\hat{\sigma}\left(S_T,T;T_s,S_T\right) = \hat{\sigma}\left(1,T;1,1\right)$ and to be consistent with the previous local volatility obtained at time $t$ we require $\hat{\sigma}\left(1,T;1,1\right) = \hat{\sigma}\left(u,T;1,1\right)$. Therefore, assuming the local volatility surface is not flat (as in the Black-Scholes case) the whole surface must move when $S$ moves. For this reason, property (iii) is called the ‘floating smile’ property. Hence static local volatility models, e.g. the original local volatility model proposed by Dupire (1994) and Derman and Kani (1994), are not scale-invariant.

3. Hedging with Scale-invariant Models

This section derives model-free properties for the price hedge ratios derived from scale-invariant option pricing models. Bates (2005) showed that if the price of an option is homogeneous of degree one in $S$ and $K$, then every scale-invariant process gives the same option delta and gamma, i.e. these sensitivities are model-free within the class of scale-invariant processes. This section extends and generalises Bates’ result as follows. First, we prove that scale invariance preserves homogeneity, in the sense that if the pay-off function is homogeneous, then the option value will also be homogeneous of the same degree at any time prior to expiry. Next, a theorem derives model-free price sensitivities of claims with pay-offs that are homogeneous of any degree $k$. 
Let \( g(t, S; K, T) \) denote the price at time \( t \geq 0 \) of a claim on \( S \) where \( S = (S_t)_{t \geq 0} \) is the underlying price at time \( t \), \( T \) is the expiry date of the claim and \( K' = (K_1, ..., K_n) \) is a set of claim characteristics in the same unit of measurement of \( S \), such as strikes and barriers. The claim may itself be a portfolio of other claims on \( S \), e.g. a straddle, butterfly spread, etc. Without loss of generality we assume the claim characteristics are known constants and we omit variables such as interest rates, dividends and other model parameters because these are of lesser importance for the hedging problem at hand. The following proposition proves that a price process is scale-invariant if and only if it preserves the homogeneity of the claim pay-off at expiry, \( G(S_T, K) \), throughout the life of the claim.

**Proposition 2.** Suppose that a pay-off at expiry \( T \) is homogeneous of degree \( k \), that is:

\[
G(uS_T, uK) = u^k G(S_T, K) \quad u \in \mathbb{R}^+
\]

then the process for \( S \) is scale-invariant if and only if

\[
g(t, uS; uK, T) = u^k g(t, S; K, T) \quad \forall t \in [0, T].
\]

**Proof.** Define the numeraire \( N_t \) so that \( Z_{t,T} = N_t / N_T \) is independent of \( S \) and \( K \). Also define the relative price \( X_{t,T} = S_T / S_t \) so that by (1) a model is scale-invariant if and only if \( X_{t,T} \) is dimensionless relative to \( S \). It follows from martingale theory (Harrison and Kreps, 1979; Harrison and Pliska, 1981) that:

\[
g(t, S; K, T) = E_{Q_N} \left[ G(S_T, K) \frac{N_t}{N_T} Z_{t,T} | \mathcal{F}_t \right] = E_{Q_N} \left[ G(S_T, X_{t,T}, K) Z_{t,T} | \mathcal{F}_t \right] \quad t \in [0, T] \tag{4}
\]

where the expectation is conditional on information up to time \( t \), denoted by \( \mathcal{F}_t \), and is under the martingale measure \( Q_N \) associated with the numeraire (see also Geman, 2005). Now apply the substitutions \( S \mapsto uS \) and \( K \mapsto uK \), and assume \( G(uS_T, uK) = u^k G(S_T, K) \). As \( Z_{t,T} \) and \( X_{t,T} \) are invariant under scaling in \( S \) and \( K \), we have

\[
g(t, uS; uK, T) = E_{Q_N} \left[ G(uS_T, X_{t,T}, uK) Z_{t,T} | \mathcal{F}_t \right]
\]

\[
= E_{Q_N} \left[ G(uS_T, uK) Z_{t,T} | \mathcal{F}_t \right]
\]

\[
= u^k E_{Q_N} \left[ G(S_T, K) Z_{t,T} | \mathcal{F}_t \right]
\]

\[
= u^k g(t, S; K, T) \quad \forall t \in [0, T]
\]

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For the converse, suppose the pay-off function is homogeneous of degree \( k \) but that the model is not scale-invariant. Then the relative price \( X_{t,T} \) is not dimensionless and scaling \( S \mapsto uS \) implies \( X_{t,T} \mapsto X_{t,T}^{u} \) where \( X_{t,T}^{u} \neq X_{t,T} \) in general. Hence, there exists at least one time \( t \) at which

\[
G(uS, X_{t,T}^{u}, uK) \neq G(uS, X_{t,T}, uK)
\]

almost surely so that, replacing into (4), we have

\[
g(t, uS; uK, T) \neq u^{k} g(t, S; K, T)
\]

and the claim price is not a homogeneous function of degree \( k \). It follows that if the claim price at every time \( t \) is a homogeneous function of degree \( k \), then the price process must be scale-invariant.

The above argument only applies to claims without the possibility of early exercise. The extension to American/Bermudan claims follows because if a European claim price is homogeneous of degree \( k \) at all times, then so is the American/Bermudan equivalent. At any time \( t \) before expiry, the claim is either exercised and its value equals the pay-off \( G(S_{t}, K) \), which is homogeneous by assumption, or not exercised and the claim value follows the same p.d.e. as the European claim, which is homogeneous for all \( t \). Thus, the American/Bermudan claim price is also homogeneous of degree \( k \) for all \( t \). Conversely, if the pay-off were homogeneous but the American/Bermudan price were not, then the European price could not be homogeneous because they are based on the same p.d.e., and the price process would not be scale invariant.

We have assumed above that the pay-off depends on the value of \( S_{T} \) at expiry only. More generally, the pay-off of path-dependent claims is a function of the whole path of \( S \), i.e. \( G\{S_{t}\}_{0\leq t \leq T}, K \). It follows that the martingale argument in Proposition 2 can be extended for path-dependent claims provided that the claim can still be replicated by a ‘mean-self-financing’ portfolio of primitive securities (see e.g., Schweizer, 1991).

Many types of options have homogeneous pay-off functions. Pay-offs that are homogeneous of degree zero include the log-contract, which pays \( \ln(S_{T}/S_{0}) \) at expiry, and a binary option, which pays \( 1_{\{S_{T}>K\}} \) for a call or \( 1_{\{K>S_{T}\}} \) for a put. Power options, for
instance those with pay-off \( \left( (S_T - K)^+ \right)^k \), are homogeneous of degree \( k > 1 \). But most claims have pay-off functions that are homogeneous of degree one, including:

- **Standard options**: e.g. a vanilla call pays \( (S_T - K)^+ \);
- **Cash-or-nothing options**: \( K1_{\{S_T>K\}} \) for a call or \( K1_{\{K>S_T\}} \) for a put;
- **Asset-or-nothing options**: \( S_T1_{\{S_T>K\}} \) for a call or \( S_T1_{\{K>S_T\}} \) for a put;
- **Look-back options**: \( (S_T - S_{\text{min}})^+ \) for a call or \( (S_{\text{min}} - S_T)^+ \) for a put;
- **Look-forward options**: \( (S_{\text{max}} - K)^+ \) for a call or \( (K - S_{\text{max}})^+ \) for a put;
- **Barrier options**: e.g. a single barrier up-and-out call pays \( (S_T - K)^+ 1_{\{S_T>K\}} \). Multiple barrier options are also homogeneous of degree one;
- **Asian options**: e.g. \( (A_T - K)^+ \) or \( (S_T - A_T)^+ \) where \( A_T \) is an average of prices prior to and at expiry;
- **Compound options**: e.g. \( C(T_1, T_2) - K \) where \( C(T_1, T_2) \) is the value of a vanilla call at time \( T_1 \), with expiry date \( T_2 > T_1 \);
- **Forward start options**: e.g. \( (S_{T_2} - S_{T_1})^+ \), where the strike is set as the at-the-money strike at \( T_1 < T_2 \). Cliquet options, which are a series of forward start options, are also homogeneous of degree one.

Proposition 2 shows that when a scale-invariant process is used to value any of the claims mentioned above, the claim price at any point in time before expiry will be homogeneous and have the same degree of homogeneity as its pay-off.

**Theorem 1.** Suppose the claim pay-off, \( G(S_T, K) \), is homogeneous of degree \( k \) and that \( S \) is generated by a scale-invariant process. Then all partial derivatives of the claim price with respect to \( S \) at any time \( t < T \) are given by linear combinations of \( g = g(t; S_T, K, T) \) and its partial derivatives with respect to \( K \), and in particular:  

\[ \frac{\partial^i g}{\partial K^j} \bigg|_{S = S_t} \]

\[ \frac{\partial^j g}{\partial K^i} \bigg|_{S = S_t} \]

Remark on Notation: In the theorem all claim prices and derivatives of these prices are functions of \( (t; S_T, K, T) \) but we have dropped this dependence for ease of notation. Also \( \left( \frac{\partial g}{\partial K} \right)_{ex} \) is the gradient vector of partial derivatives and \( \left( \frac{\partial^2 g}{\partial K^2} \right)_{ex} \) is the Hessian matrix of second partial derivatives of \( g \) with respect to \( K \), all evaluated at time \( t \) when \( S = S_t \). Finally \( K' \) denotes the transpose of \( K \).
\[ g_S = S^{-1} (k g - K_{g_k}^r ) \]
\[ g_{S\bar{S}} = S^{-2} \left[ K_{g_k}^r g_k + (k - 1) (k g - 2K_{g_k}^r) \right] \]  

**Proof.** Since \( S \) is generated by a scale-invariant process, Proposition 2 yields:
\[ g(t, uS; uK, T) = u^kg(t, S; K, T) \quad \forall t \in [0, T]. \]  
Differentiating (6) with respect to \( u \) and setting \( u = 1 \) we obtain:
\[ S g_S (t, S; K, T) + K_{g_k}^r (t, S; K, T) = k g(t, S; K, T) \]  
which is the well-known Euler’s theorem for homogeneous functions (Leonhard Euler, 1707 – 1783). After re-arranging, this gives the expression for \( g_S \) in (5). For \( g_{SS} \), we differentiate (6) twice with respect to \( u \) and set \( u = 1 \) to obtain:
\[ S^2 g_{S\bar{S}} + 2S K_{g_k}^r + K_{g_k}^r K = k (k - 1) g. \]  
On differentiating (7) with respect to \( S \) we obtain:
\[ K_{g_k}^r = (k - 1) g_S - S g_{S\bar{S}}. \]  
Combining (7), (8) and (9) gives the expression for \( g_{SS} \) in the theorem.

Now assume \( g_{S^m} = \sum_{i=0}^{m} A_i g_k^r B_i \) for \( m \geq 1 \), where \( g_{S^m} \) denotes the \( m \)-th partial derivative of \( g \) with respect to \( S \) and \( \left( g_k^r \right)_i \) is the \( i \)-dimensional matrix of \( i \)-th partial derivatives of \( g \) with respect to \( K \), and in particular we define \( g_k^r = g \). \( A_i (S, K) \) and \( B_i (S, K) \) are known matrices at time \( t \). It follows that
\[ g_{S^{m+1}} = \left( g_{S^m} \right)_S = \sum_{i=0}^{m} \left[ (A_i)_S g_k^r B_i + A_i \left( g_k^r \right)_S B_i + A_i g_k^r \left( B_i \right)_S \right] \]
where
\[ \left( g_k^r \right)_S = (g_S)_S = S^{-1} (k g - K_{g_k}^r ) = S^{-1} \left( (k - 1) g_k^r - K_{g_{k+1}}^r \right) \]
so that we may write \( g_{S^{m+1}} = \sum_{i=0}^{m+1} \tilde{A}_i g_k^r \tilde{B}_i \) for some matrices \( \tilde{A}_i (S, K) \) and \( \tilde{B}_i (S, K) \). As \( m \) is arbitrary, we conclude that all partial derivatives with respect to \( S \) are linear combinations of \( g_k^r \). \( \square \)

The theorem implies that if several prices of claims of the same type are observable in the market, then so are the price hedge ratios of these claims. Consider first the case of standard European options. These are highly liquid contracts and options prices for a wide
range of strikes and expiry dates are available. Therefore, if two scale-invariant models are calibrated to the same market prices of calls and puts, the theorem implies that both models should give the same delta and the same gamma for the options, a result that was also proved by Bates (2005). Empirically, there will be differences between the delta and gamma obtained using the two models but this is due to the fact that the models do not fit market data equally well. On the other hand, suppose a model is calibrated to standard European calls and puts and then it is used to price and hedge cliquet options. Because the price and the claim characteristic sensitivities of the cliquet will be model dependent, so will be the price hedge ratios.

In summary, any two scale-invariant models yield the same price hedge ratios for a claim with homogeneous pay-off and characteristics $K$ given that prices of claims of the same type and with characteristics in the neighbourhood of $K$ are used to calibrate the models and that both models fit these prices exactly. A perfect fit is normally not attainable in practice, but if two scale-invariant models fit the data reasonably well then no significant difference between the empirical hedging performances of the models should be observed.

### 3.1. Hedging Standard European Options

The Black-Scholes model is scale-invariant, but it is not smile-consistent. It cannot explain the observed prices for standard calls and puts and it produces partial derivatives with respect to the strike $K$ that are different from those empirically observed in the market. As a result, using (5) we conclude that there is a difference between the deltas and gammas of smile-consistent scale-invariant models and the Black-Scholes delta and gamma for standard European options.\(^4\)

We now use property (ii) of Proposition 1 to derive the theoretical relationship between smile-consistent and Black-Scholes deltas and gammas. Differentiating (2) with respect to $S$\(^4\) Remark on Notation: In the following, the implied volatility and its derivatives are functions of $(K,T;t,S)$ but we have again dropped this dependence for ease of notation. We shall use the notations $\delta^w$, $\gamma^w$, $\delta^b$ and $\gamma^b$ to denote the delta and gamma of a standard European option from a scale-invariant smile-consistent (SISC) model and from the Black-Scholes model, respectively. That is, $\delta^w = f_3^w$, $\gamma^w = f_3^w$, $\delta^b = f_3^b$ and $\gamma^b = f_3^b$ where $f(t,S,K,T)$ and $f^b(t,S,K,T,\theta)$ are the prices of the standard European option under the smile-consistent model and under the Black-Scholes model respectively (c.f. $g(t,S,K,T)$, which is used for the price of a general claim). We also use the notation $\psi = f_0^b$ and $\chi = f^b$ to denote the first and second order sensitivities of the Black-Scholes price with respect to implied volatility, which we call ‘vega’ and ‘kappa’.
and using (3) gives:

\[
\delta^v(t, S; K, T) = \delta^b(t, S; K, T, \theta) - \nu^b(t, S; K, T, \theta) \theta_k \frac{K}{S}
\]  

(10)

and differentiating (10) again with respect to \(S\) yields:

\[
\gamma^v(t, S; K, T) = \gamma^b + \left( \nu^b \theta_{kk} + 2\nu^b \theta_k + \nu^b \theta_k \theta_k \right) \left( \frac{K}{S} \right)^2
\]  

(11)

where the Black-Scholes hedge ratios are functions of \((t, S; K, T, \theta)\) as in (10). Note that (10) and (11) are model-free because \(\theta_k\) and \(\theta_{kk}\) are observable. They are also given by (5) when \(g = f\).

The relationship (10) between the smile-consistent delta and the Black-Scholes delta shows that the difference between the two depends on the Black-Scholes vega and the moneyness of the option as given by \(K/S\), both of which are positive, and on the slope of the implied volatility in the strike metric. In a typical equity or equity index option, the slope of the implied volatility is negative, so the smile-consistent delta will be greater than the Black-Scholes delta, except perhaps for options with very high strikes. Figure 1a shows the smile-consistent delta and the Black-Scholes delta for the S&P 500 index options with expiry at 15th June 2004, plotted as a function of moneyness. The date was 21st May 2004 and prices for 34 different strikes were available. The smile-consistent delta was considerably greater than the Black-Scholes delta, except at very high strikes.

The relationship (11) between the gamma of a scale-invariant smile-consistent (SISC) model and the Black-Scholes gamma shows that the difference between the two depends on the Black-Scholes vega, its strike sensitivity, the Black-Scholes kappa, the moneyness of the option, and the slope and convexity of the implied volatility in the strike metric. Thus the difference could be positive or negative. Figure 1b shows the smile-consistent gamma and the Black-Scholes gamma for the same S&P 500 index options as in figure 1a. Both the low strike options and the very high strike options have smile-consistent gammas that are less than the Black-Scholes gamma, whereas high strike options and those with strikes that are close to at-the-money have smile-consistent gammas that are greater than the Black-Scholes gamma.

Figure 1 about here
4. Minimum Variance Hedge Ratios

So far we have defined delta and gamma as the usual partial derivatives of the claim price with respect to the underlying price. However, when there are extra dynamic features in the model such as stochastic volatility or stochastic interest rates, it turns out that these might not be the most efficient hedge ratios to use in a delta or delta-gamma hedging strategy. We define the minimum variance delta, $\delta_{mv}$, as the amount of the underlying asset at time $t$ that reduces the instantaneous covariance of a delta-hedged portfolio, $\Pi = g - \delta_{mv}S$, with the underlying asset price $S$ to zero. That is,

$\langle d\Pi, dS \rangle = \langle dg - \delta_{mv} dS, dS \rangle = \langle dg, dS \rangle - \delta_{mv} \langle dS, dS \rangle = 0$.

As before, we drop the dependence of $\Pi, g$ and $\delta_{mv}$ on $(t, S; K, T)$ for ease of notation.

In the Black-Scholes framework, the minimum variance delta is the same as the first partial derivative of the claim price with respect to $S$, but this is not the case when any model component such as the volatility or interest rates is correlated with the asset price. Suppose the spot volatility (or variance) is a continuous and stochastic process itself and there are no jumps. Then the dynamics of the claim price $g = g^m(t, S; \sigma; K, T)$ are given by Itô’s formula as:

$$dg = g_s dt + g_s dS + g_{ss} d\sigma + \frac{1}{2} g_{ss} dS^2 + \frac{1}{2} g_{s\sigma} d\sigma^2 + g_{ss} d\sigma dS$$

where the subscripts of $g$ denote partial differentiation and the quadratic terms are adapted processes of order $dt$.

Therefore, in a stochastic volatility model without jumps, the minimum variance hedge ratio is given by the ratio of the instantaneous covariance between increments in the claim price and the underlying price and the instantaneous variance of the increments in the underlying price. That is:

$$\delta_{mv}(t, S; \sigma; K, T) = \frac{\langle dg, dS \rangle}{\langle dS, dS \rangle} = \frac{\langle g_s dt + g_{ss} dS + g_{s\sigma} d\sigma, dS \rangle}{\langle dS, dS \rangle} = g_s + g_{ss} \frac{\langle d\sigma, dS \rangle}{\langle dS, dS \rangle}.$$

Intuitively, this resembles a total derivative of the claim price with respect to $S$, in which

---

5 This is also known as local risk minimization, and has been studied extensively in the context of incomplete markets by Schweizer (1991), Bakshi et al. (1997), Frey (1997), Heath et al. (2001) and others.
the total derivatives $\frac{dg}{dS}$ and $\frac{d\sigma}{dS}$ are defined in expectation as:

$$
\frac{dg}{dS} = \langle dg, dS \rangle \quad \text{and} \quad \frac{d\sigma}{dS} = \langle d\sigma, dS \rangle.
$$

Therefore,

$$
\delta^w_{mv} = \frac{dg}{dS} = g_s + g_s \frac{d\sigma}{dS}.
$$

(12)

The minimum variance delta is the standard delta, $g_s$, plus an additional term that is non-zero when the two Brownian motions driving price and the volatility are correlated. We see that in stochastic volatility models with uncorrelated Brownian motions (such as Hull and White, 1987; Stein and Stein, 1991; Nelson, 1990; and others) the minimum variance delta is equal to the standard delta; and if these models are also scale-invariant, the minimum variance delta is given by (5). Otherwise, in stochastic volatility models with non-zero correlation (such as Heston, 1993, and Hagan et al., 2002) the minimum variance delta is not equal to the standard delta.

The minimum variance gamma can be derived by setting

$$
\delta^w_{mv}(t,S_t; \sigma; K, T) = 0 \Rightarrow \gamma^w_{mv} = \frac{\langle d\delta^w_{mv}, dS \rangle}{\langle dS, dS \rangle}
$$

and applying Itô’s formula to $\delta^w_{mv}(t,S_t; \sigma; K, T)$ to obtain:

$$
\gamma^w_{mv} = \frac{d^2 g}{dS^2} = \left(\delta^w_{mv}\right)_g + \left(\delta^w_{mv}\right)_\sigma \frac{d\sigma}{dS} = g_{ss} + \left(2g_{so} \frac{d\sigma}{dS} + g_{ss} \left(\frac{d\sigma}{dS}\right)^2 + g_{oo} \frac{d^2 \sigma}{dS^2}\right).
$$

(13)

where the second-order total derivative in the right-hand side is interpreted as

$$
\frac{d^2 \sigma}{dS^2} = \left(\frac{d\sigma}{dS}\right)_s + \left(\frac{d\sigma}{dS}\right)_o \frac{d\sigma}{dS}.
$$

Hence in stochastic volatility models with zero price-volatility correlation the minimum variance gamma is equal to the standard gamma and if the model is also scale-invariant, the minimum variance gamma is the model-free gamma in (5). Otherwise, the minimum variance gamma can be greater than or less than the standard gamma.

In general, the minimum variance delta and gamma account for the total effect of a change in the underlying price, including the indirect effect of the price change on the claim price via its effect on the volatility (or any other parameter that is correlated with the underlying price).
We now consider the minimum variance hedge ratios for local volatility models, scale-invariant and otherwise. In the stochastic volatility case, we used the second source of randomness from the volatility process to motivate an adjustment to the hedge ratios; but in local volatility models there is just one source of randomness. Nevertheless, because the instantaneous volatility $\sigma(t,S)$ in a local volatility model is a function of $S$, it is also a continuous process and it has dynamics given by Itô’s formula as:

$$d\sigma = \left(\sigma_t + \frac{1}{2}\sigma^2 S^2 \sigma_{SS}\right) dt + \sigma_S dS$$

which can be interpreted as a stochastic volatility model with perfect correlation between the instantaneous volatility and the underlying asset price.

Therefore, using (12) and (13), the minimum variance local volatility hedge ratios are:

$$\delta^{lv}_{mw} = g_s + g_{\sigma_s} \sigma_s$$

$$\gamma^{lv}_{mw} = g_{SS} + \left(2g_{\sigma_s}\sigma_s + g_{\sigma_{SS}}\sigma_{SS}\right)$$

If the instantaneous volatility is an explicit parameter of the model the partial derivatives $g_{\sigma_s}$, $g_{\sigma_{SS}}$ and $g_{\sigma_{SS}}$ are well-defined. Otherwise it may be possible to re-parameterize the model in terms of this. See the scale-invariant version of the CEV model below, for example.

We conclude this section by deriving the minimum variance hedge ratios for some specific models, viz. the Heston (1993) stochastic volatility model, the constant elasticity of variance (CEV) model (Cox, 1975), a scale-invariant modification of this model, and the SABR model of Hagan et al. (2002). We have chosen these models to represent both scale-invariance (the Heston model) and non-scale-invariance (the SABR model). We use the CEV model to illustrate how a non-scale-invariant model may be easily converted into a scale-invariant form. The empirical hedging performance of these models, using both minimum variance and standard deltas and gammas, will be tested in the next section.

4.1. Heston’s stochastic volatility model

The Heston (1993) model:

$$\frac{dS}{S} = \mu dt + \sqrt{V} dW$$

$$dV = a(m-V) dt + b\sqrt{V} dZ \quad \langle dW, dZ \rangle = \varrho dt$$

is scale-invariant. Hence:
The above emphasises that the only model-dependent part of the hedge ratio is the second term in the right-hand side.

4.2. The CEV model

The dynamics of the underlying price in the CEV model, introduced by Cox (1975), are:

\[
\frac{dS}{S} = \mu dt + \alpha S^\beta dW
\]

where \( \alpha > 0 \) and \( \beta < 0 \). The CEV model above is clearly not scale-invariant because \( S \) appears in the right-hand side of the price diffusion. However, on fixing \( \beta = 0 \), the CEV model may be written in the following scale-invariant form:

\[
\frac{dS}{S} = \mu dt + \sigma(t, S; \alpha, \sigma_0) dW \quad \text{where} \quad \sigma(t, S; \alpha, \sigma_0) = \sigma_0 \left( \frac{S}{S_0} \right)^\beta.
\]

This model is equivalent to the original formulation in the sense that the two models produce exactly the same prices. The minimum variance hedge ratios of the scale-invariant form of the CEV model are:

\[
\delta_{\text{mse}} = \delta_{\mu} + g_n \frac{\sigma^2}{S},
\]

\[
\gamma_{\text{mse}} = \gamma_{\mu} + \frac{\sigma^2}{S} \left( \frac{\sigma^2}{S^n} + 2 g_n \frac{\beta - 1}{S} \right).
\]

It is simple to verify that the minimum variance delta of the scale-invariant CEV model is equal to the standard delta of the original CEV model (and similarly for gammas): since

\( g_{\text{mse}}(t, S; K, T, \alpha, \beta) = g_{\text{mse}}(t, S; K, T, \alpha, \beta) \)

we have

\( g_{\text{mse}}(t, S; K, T, \alpha, \beta) = g_{\text{mse}}(t, S; K, T, \alpha, \beta) + g_{\text{mse}}(t, S; K, T, \alpha, \beta) \alpha \)

and the right-hand-side above is exactly the minimum variance delta in (17). We conclude that no adjustment is necessary to make the standard delta and gamma of the original CEV model into minimum variance hedge ratios. In fact, this argument may be extended to other ‘sticky-tree’ local volatility models. That is, the minimum variance hedge ratios for...
sticky-tree local volatility models are the same as the standard delta and gamma given by
the partial derivatives \( g_\delta \) and \( g_{\delta\delta} \).

4.3. The SABR model
The ‘stochastic-\( \alpha\beta\nu \)’ model of Hagan et al. (2002) has recently become popular amongst
practitioners. The model takes the CEV functional form for the dynamics of the forward
price \( F \) and allows the alpha parameter to be driven by a correlated diffusion as follows:
\[
\begin{align*}
    dF &= \alpha F^\beta dW' \\
    d\alpha &= \rho \nu d\alpha + \sqrt{\rho \nu} dZ \\
    \langle dW', d\alpha \rangle &= \rho dt
\end{align*}
\]
(18)

The model is not scale-invariant unless \( \beta = 1 \). The minimum variance delta and gamma
(with respect to \( F \)) of a claim whose price is \( g = g^{\text{sabr}}(t, F; K, T, \alpha, \beta, \nu, \rho) \) are given by:
\[
\begin{align*}
    \gamma^{\text{sabr}}_\delta &= \frac{dg}{dF} = \left\langle \frac{dg}{dF}, dF \right\rangle = g_F + g_{\alpha} \frac{\nu \rho}{F^\beta} \\
    \gamma^{\text{sabr}}_{\delta\delta} &= \frac{d^2g}{d\delta^2} = g_{FF} + \frac{\nu \rho}{F^\beta} \left( \frac{\nu \rho}{F^\beta} \right) g_{\alpha\alpha} + 2 g_{F\alpha} - \frac{\beta}{F^\beta} g_{\alpha} \\
\end{align*}
\]
(19)

On the right-hand-side, \( g_F \) and \( g_{FF} \) are the standard delta and the standard gamma of the
SABR model with respect to \( F \). They are not model-free because the model is not scale-
invariant. The second term in (19) captures the correlation between \( F \) and \( \alpha \), as in other
stochastic volatility models.

This example is illustrative since the SABR model is not scale-invariant but it still requires
an adjustment to the standard delta and gamma to obtain minimum variance hedge ratios.
In the other non-scale-invariant model that we consider, the CEV model, this adjustment
was not necessary, as shown above.

5. Empirical Results
This section compares the hedging performance of the option pricing models considered
above using both standard delta and gamma and the minimum variance hedge ratios. We
restrict the study to these models because on testing the model-free hedge ratios from
several different scale-invariant smile-consistent models no significant difference between
the model’s performances was found. We have therefore used the Heston (1993) model as
a representative scale-invariant smile-consistent (SISC) model. Its delta and gamma are
model-free and given by (5) but since the price-volatility correlation is non-zero, the minimum variance hedge ratios (16) will be different from the model-free hedge ratios. The CEV and SABR models were included in our study because they have the potential to generate significantly different results. The CEV model can be parameterised as either scale-invariant or non-scale-invariant. In its scale-invariant form, the standard price hedge ratios are model-free, but the minimum variance hedge ratios are different – in fact, these are equal to the standard price hedge ratios of the non-scale-invariant form of the model, as shown above. Finally, the SABR model is not scale-invariant, thus its price hedge ratios are not model-free and, since the price-volatility correlation is non-zero, the minimum variance hedge ratios will also be different from the standard price hedge ratios. For the SABR hedges we have set $\beta = 0$.

Bloomberg data on the June 2004 European call options on the S&P 500 index, i.e. daily close prices from 02 Jan 2004 to 15 June 2004 (111 business days) for 34 different strikes (from 1005 to 1200), have been applied in this study. Only the strikes within \( \pm 10\% \) of the current index level were used for the model's calibration each day but all strikes were used for the hedging strategies. The delta hedge strategy consists of one delta-hedged short call in each option, rebalanced daily. That is, one call on each of the 34 strikes from 1005 to 1200 is sold on 16\(^{th}\) January (or when the option is issued, if later than this) and hedged by buying an amount $\delta$ (delta) of the underlying asset, where $\delta$ is determined by both the model and the option's characteristics. The portfolio is rebalanced daily, assuming zero transaction costs, stopping on 2\(^{nd}\) June because from then until the expiry date the fit to the smile worsened considerably for most of the models. The delta-gamma hedge strategy again consists of a short call in each option, but this time an amount of the 1125 option, which is closest to at-the-money in general over the period, is bought. This way the gamma on each option is set to zero and then we delta hedge the portfolio as above. This option-by-option strategy on a large and complete database of liquid options allows one to assess the effectiveness of hedging by strike or moneyness of the option, and day-by-day as well as over the whole period. A data set of P&L (profit and loss) with 1324 observations is obtained.

Each model was calibrated daily by minimizing the root-mean-square-error between the model implied volatilities and the market implied volatilities of the options used in the calibration set. For the Black-Scholes model, the deltas and gammas are obtained directly
from the market data and there is no need for model calibrations. We used the closed-form solution for the Heston model based on Fourier transforms (Lewis, 2000), chose a volatility risk premium of zero and set the long-term volatility at 12%. Finally, the calculation of the CEV hedge ratios is based on the non-central chi-square distribution result of Schroder (1989), and SABR hedge ratios are computed with the analytical approximation of the implied volatility of Hagan et al. (2002).

The deltas and gammas of each model, whilst changing daily, exhibit some strong patterns when they are plotted by strike or by moneyness: the same shapes emerge day after day. In figures 2a and 2b, we compare the deltas and gammas from the different models on 21st May 2004, a day exhibiting typical patterns for the models’ delta and gamma of S&P 500 call options. In figure 2a, the partial price derivative that is common to all SISC models produces a delta that is greater than the Black-Scholes delta for all but the very high strikes, as we have already seen in figure 1a. The SABR model delta lies between the Black-Scholes and SISC deltas. So if the Black-Scholes model over-hedges in presence of the skew (as shown by Coleman et al., 2001) then both scale-invariant and SABR deltas should perform worse than the Black-Scholes model. A different picture emerges when minimum variance hedge ratios are used. In the CEV and SABR models (which are not scale-invariant) and in the Heston model (which is scale-invariant) the minimum variance deltas are generally lower than the Black-Scholes deltas. Another pattern is observed in figure 2b for the gammas. SISC and SABR gammas are lower than the Black-Scholes gamma for in-the-money calls and greater than the Black-Scholes gamma for out-of-the-money calls (except for very deep out-of-the-money calls) while the opposite is observed when minimum variance gammas are considered. So partial price sensitivities will under-hedge/over-hedge the gamma risk for in-the-money/out-of-the-money calls respectively, relative to the Black-Scholes hedges.

Table 1 reports the sample statistics of the aggregate daily P&L for each model, over all options and over all days in the hedging period. The models are ordered by the standard deviation of the daily P&L. Small skewness and excess kurtosis in the P&L distribution are also desirable – high values for these sample statistics indicate that the model was spectacularly wrong on a few days in the sample. Another important performance criterion is that the P&L be uncorrelated with the underlying asset. In our case, over-hedging would
result in a significant positive correlation between the hedge portfolio and the S&P 500 index return. We have therefore performed a regression, based on all 1324 P&L data points, where the P&L for each option is explained by a quadratic function of the S&P 500 returns. The lower the $R^2$ from this regression, reported in the last column, the more effective the hedge.

Table 1 about here

According to these criteria, the best delta hedges are obtained from the minimum variance hedge ratios, irrespective of the underlying model used. The minimum variance deltas yield lower standard deviations than the Black-Scholes delta, and these also have P&L that are closest to being normally distributed according to the observed skewness and excess kurtosis. Conversely, both SABR and SISC deltas perform worse than the Black-Scholes delta. Apart from this, the positive mean P&L for delta hedging is a result of the short volatility exposure and gamma effects, since we have only rebalanced daily. The delta-gamma hedging results in part (b) of Table 1 show a mean P&L that is close to zero. On adding a gamma hedge it is remarkable that the Black-Scholes model performance improves considerably, whilst the other models ranked more or less as before. Also notable is that the SABR model minimum variance hedge has the smallest $R^2$ in both tables.

One possible explanation for the superiority of the Black-Scholes model in Table 1b is that the same hedging strategy is used to gamma hedge or vega hedge vanilla options: the ratio of the gammas is equal to the ratio of the vegas in the Black-Scholes model. This is evidence that most of the imperfections of the Black-Scholes model can be dealt with by hedging the movements in implied volatility. In fact, Bakshi, Cao and Chen (1997) also find that vega hedging with the Black-Scholes model performs well except for low strike in-the-money call options. These authors also show that once stochastic volatility is modelled, the inclusion of jumps leads to no discernable improvement in hedging performance, at least when the hedge is rebalanced frequently, because the likelihood of a jump during the hedging period is too small. They also find that the inclusion of stochastic interest rates can improve the hedging of long-dated OTM options, but for other options stochastic volatility is the most important factor to model.

Results on hedged portfolio P&L standard deviation by moneyness, averaged over all days in our sample are given in Table 2. This table shows that the apparent superiority of the
Black-Scholes model for delta-gamma hedging is due to its success at hedging the strikes slightly higher than at-the-money. This may be linked to our finding in figure 2 that the Black-Scholes gamma is similar to the minimum variance gammas for near-the-money options. For out-of-the-money calls, the minimum variance hedge ratios from the Heston model give the lowest standard deviation of hedged portfolio P&L. Hedging performance is particularly bad when the standard hedge ratios for SABR and SISC models are used.

Table 2 and Figure 3 about here

Figures 3a and 3b plot the cumulative distribution functions of the hedging P&L, taken over all options and over all days in the sample. Figure 3a depicts the P&L from delta hedging only and Figure 3b depicts the P&L from delta-gamma hedging. In both charts, there are two distinct groups: the minimum variance hedging strategies (CEV, Heston (MV) and SABR (MV)) and the standard hedging strategies (SABR and SISC). The former group is more efficient because it produces a P&L distribution that is less dispersed around the mean. The Black-Scholes model lies in between the two groups in (a) and very close to the minimum variance hedges in (b). The P&L for delta-gamma hedging with SABR and SISC models are also slightly shifted to the right. These findings are consistent with Table 1, which reports the moments of the same distributions.

Applying a Kolmogorov-Smirnoff test (Massey, 1951; Siegel, 1988) to these distribution functions yields the results in Table 3. The null hypothesis is that the two P&L distributions are the same and the Kolmogorov-Smirnoff statistic is asymptotically \( \chi^2 \) distributed with 2 degrees of freedom. Significant values at the 10%, 5% or 1% levels are marked with one, two or three asterisks, respectively. The results confirm our theoretical findings. There are very significant differences between the P&L from minimum variance deltas and gammas (CEV, Heston (MV) and SABR (MV)) and the P&L from standard deltas and gammas (SISC and SABR). However no significant difference is found between different minimum variance strategies for delta hedging, and similarly for delta-gamma hedging. Any one of the three models considered provides an effective delta or delta-gamma hedge for SP 500 call options. Finally, the differences between the BS P&L and the P&L from the minimum variance hedge ratios are significant for delta hedging but not for delta-gamma hedging.

Table 3 about here
The similarity in the performance of minimum variance hedges is certainly intriguing as these hedge ratios were not expected to be model-free. Since all three models have been calibrated to the same implied volatility smile we do expect them to produce roughly the same local volatility surface at the calibration time, as follows from the forward equation (Dupire, 1996; Derman and Kani, 1998). Yet each model assumes different underlying price dynamics, so both the option price and the local volatility dynamics will differ from one model to another. Thus it is not intuitively obvious why the minimum variance hedge ratios should be the same for all three models. If true, this would add an important constraint to the permissible dynamics of local volatility, a result that is left to further research.

6. Conclusions

Merton (1973) was probably to first to identify that level-independent asset returns lead to the homogeneity of vanilla option prices. More recently Bates (2005) proved that scale invariance also implies that vanilla option price sensitivities are model-free. Both authors argue that scale invariance is a natural and intuitive property to require for models that price options on financial assets. Yet these authors examined a limited set of models, applied only to vanilla options. Moreover, Bates did not consider the optimality of option price partial derivatives as hedge ratios.

Starting from a more general point of view we have proved that scale invariance preserves the homogeneity of a contingent claim pay-off throughout the life of the claim. In fact, for any claim with homogeneous pay-off, a model is scale-invariant if and only if the claim price is homogeneous at all times. We use this property to prove that all partial derivatives of the claim price with respect to the underlying price are given by linear combinations of the claim price and its derivatives with respect to the claim characteristics. Thus scale invariance implies that price hedge ratios will be model-free for any claim with a homogeneous pay-off and claim prices that are observable in the market. This generalises Bates’ result to any claim with homogeneous pay-off.

We then showed how minimum variance hedge ratios require an adjustment to the model-free delta and gamma of scale-invariant models whenever there is a non-zero correlation between the underlying price and any other stochastic component of the model. Empirical results on S&P 500 index options showed that, whilst the standard (model-free) hedge
ratios of scale-invariant models perform worse than the Black-Scholes model, minimum variance hedge ratios provide better hedges on average. Our results also reveal a remarkable similarity in the performance of minimum variance hedges, indicating that some model-free relationship may hold even for minimum variance hedge ratios.

There remains much scope for theoretical research arising from the results in this paper: we have restricted the present study to continuous semi-martingales and Markovian price processes but an extension to general semi-martingales is possible; and the behaviour of scale-invariant models under other hedging strategies, such as super-hedging, utility maximization or mean-variance hedging, remains to be explored.

References


Fig. 1. S&P 500 smile-consistent and Black-Scholes hedge ratios by moneyness on May 21st 2004: Figure (a) compares the ‘model-free’ delta from a scale-invariant smile-consistent (SISC) model and the Black-Scholes (BS) delta on the same date and on the same options. Figure (b) shows the corresponding gammas. In each figure, the hedge ratios are drawn as functions of \( K/S \).
Fig. 2. The models’ delta and gamma by moneyness on May 21st 2004: Figure (a) shows the standard and minimum variance (MV) delta of the Heston and SABR model, the SISC model-free delta, and the deltas of the CEV and BS models (for which the standard deltas are also MV). Figure (b) shows the corresponding gammas. In each figure, the hedge ratios are drawn as functions of $K/S$ and May 21st was chosen as a day when all the hedge ratios exhibited their typical pattern.
Fig. 3. Cumulative distribution functions of the hedging P&L, taken over all options and over all days in the sample: In both charts there are two distinct groups: the minimum variance hedging strategies (CEV, Heston (MV) and SABR (MV)) and the non-MV hedging strategies (SABR and SISC). The former group is more efficient because it produces a P&L that is less dispersed. The BS model lies in between the two groups in (a) and very close to the minimum variance hedges in (b).
Table 1
Sample Statistics of the Aggregate Daily P&L for Delta Hedging

### a. Delta Hedging

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CEV</td>
<td>0.1462</td>
<td>0.5847</td>
<td>-0.3424</td>
<td>0.7820</td>
<td>0.113</td>
</tr>
<tr>
<td>SABR (MV)</td>
<td>0.1218</td>
<td>0.6080</td>
<td>-0.4040</td>
<td>0.8243</td>
<td>0.109</td>
</tr>
<tr>
<td>Heston (MV)</td>
<td>0.1370</td>
<td>0.6103</td>
<td>-0.5704</td>
<td>1.6737</td>
<td>0.152</td>
</tr>
<tr>
<td>BS</td>
<td>0.1401</td>
<td>0.7451</td>
<td>-0.7029</td>
<td>2.0370</td>
<td>0.412</td>
</tr>
<tr>
<td>SABR</td>
<td>0.1427</td>
<td>0.9948</td>
<td>-0.6485</td>
<td>1.7099</td>
<td>0.629</td>
</tr>
<tr>
<td>SISC</td>
<td>0.1373</td>
<td>1.1788</td>
<td>-0.5928</td>
<td>1.4834</td>
<td>0.693</td>
</tr>
</tbody>
</table>

### b. Delta-Gamma Hedging

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>-0.0014</td>
<td>0.2612</td>
<td>-0.4353</td>
<td>2.5297</td>
<td>0.020</td>
</tr>
<tr>
<td>CEV</td>
<td>0.0098</td>
<td>0.2691</td>
<td>-0.0291</td>
<td>3.0850</td>
<td>0.051</td>
</tr>
<tr>
<td>Heston (MV)</td>
<td>0.0111</td>
<td>0.2789</td>
<td>0.1929</td>
<td>3.6019</td>
<td>0.029</td>
</tr>
<tr>
<td>SABR(MV)</td>
<td>0.0044</td>
<td>0.3045</td>
<td>-0.3003</td>
<td>3.0032</td>
<td>0.016</td>
</tr>
<tr>
<td>SABR</td>
<td>0.0289</td>
<td>0.3821</td>
<td>-0.4845</td>
<td>5.0482</td>
<td>0.057</td>
</tr>
<tr>
<td>SISC</td>
<td>0.0428</td>
<td>0.4548</td>
<td>0.0208</td>
<td>4.0123</td>
<td>0.060</td>
</tr>
</tbody>
</table>

This table reports the sample statistics of the aggregate daily P&L for each model, over all options and over all days in the hedging period, for the delta and delta-gamma hedging strategies with daily rebalancing. The models are ordered by the standard deviation of the daily P&L. Small skewness and excess kurtosis are desirable. We also performed a regression, based on all 1324 P&L data points, where the P&L for each option is explained by a quadratic function of the S&P 500 returns. The $R^2$ from this regression, reported in the last column of the table, is small when the hedge is effective.
Table 2

Standard Deviation of the Daily P&L Aggregated by Moneyness of Option

a. Delta Hedging

<table>
<thead>
<tr>
<th>K/S</th>
<th>0.90-0.95</th>
<th>0.95-1.00</th>
<th>1.00-1.05</th>
<th>1.05-1.10</th>
<th>1.10-1.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best</td>
<td>SABR (MV)</td>
<td>0.3657</td>
<td>CEV</td>
<td>0.5740</td>
<td>CEV</td>
</tr>
<tr>
<td></td>
<td>Heston (MV)</td>
<td>0.3714</td>
<td>SABR (MV)</td>
<td>0.5988</td>
<td>Heston (MV)</td>
</tr>
<tr>
<td></td>
<td>CEV</td>
<td>0.3854</td>
<td>Heston (MV)</td>
<td>0.6161</td>
<td>SABR (MV)</td>
</tr>
<tr>
<td></td>
<td>BS</td>
<td>0.5652</td>
<td>BS</td>
<td>0.7876</td>
<td>BS</td>
</tr>
<tr>
<td>Worst</td>
<td>SISC</td>
<td>0.7357</td>
<td>SISC</td>
<td>1.2055</td>
<td>SISC</td>
</tr>
</tbody>
</table>

b. Delta-Gamma Hedging

<table>
<thead>
<tr>
<th>K/S</th>
<th>0.90-0.95</th>
<th>0.95-1.00</th>
<th>1.00-1.05</th>
<th>1.05-1.10</th>
<th>1.10-1.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best</td>
<td>Heston (MV)</td>
<td>0.1801</td>
<td>CEV</td>
<td>0.2358</td>
<td>BS</td>
</tr>
<tr>
<td></td>
<td>CEV</td>
<td>0.1853</td>
<td>SABR (MV)</td>
<td>0.2431</td>
<td>CEV</td>
</tr>
<tr>
<td></td>
<td>SABR (MV)</td>
<td>0.1984</td>
<td>BS</td>
<td>0.2561</td>
<td>Heston (MV)</td>
</tr>
<tr>
<td></td>
<td>BS</td>
<td>0.2012</td>
<td>Heston (MV)</td>
<td>0.2594</td>
<td>SABR (MV)</td>
</tr>
<tr>
<td>Worst</td>
<td>SISC</td>
<td>0.2187</td>
<td>SABR</td>
<td>0.3202</td>
<td>SABR</td>
</tr>
<tr>
<td></td>
<td>SISC</td>
<td>0.3214</td>
<td>SISC</td>
<td>0.3695</td>
<td>SISC</td>
</tr>
</tbody>
</table>

This table reports the standard deviation of daily P&L, for each model, aggregated over all options of a given moneyness and over all days in the hedging period, for the delta and delta-gamma hedging strategies, with daily rebalancing. According to this criterion, the BS model performs best only for the delta-gamma hedging of near ATM options.
### Table 3

**Kolmogorov-Smirnoff Test Results**

#### a. Delta Hedge P&L c.d.f.

<table>
<thead>
<tr>
<th></th>
<th>BS</th>
<th>SISC</th>
<th>CEV</th>
<th>Heston (MV)</th>
<th>SABR</th>
<th>SABR (MV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>-</td>
<td>29.889***</td>
<td>5.114*</td>
<td>4.923*</td>
<td>12.726***</td>
<td>6.137**</td>
</tr>
<tr>
<td>SISC</td>
<td>29.836***</td>
<td>-</td>
<td>52.664***</td>
<td>51.297***</td>
<td>5.630*</td>
<td>53.415***</td>
</tr>
<tr>
<td>CEV</td>
<td>5.153*</td>
<td>52.773***</td>
<td>-</td>
<td>1.232</td>
<td>27.968***</td>
<td>2.462</td>
</tr>
<tr>
<td>Heston (MV)</td>
<td>4.919*</td>
<td>51.253***</td>
<td>1.217</td>
<td>-</td>
<td>29.681***</td>
<td>1.666</td>
</tr>
<tr>
<td>SABR</td>
<td>12.771***</td>
<td>5.630*</td>
<td>27.978***</td>
<td>29.659***</td>
<td>-</td>
<td>28.343***</td>
</tr>
<tr>
<td>SABR (MV)</td>
<td>6.169**</td>
<td>53.458***</td>
<td>2.463</td>
<td>1.672</td>
<td>28.232***</td>
<td>-</td>
</tr>
</tbody>
</table>

This table reports Kolmogorov-Smirnoff statistics for the null hypothesis that two P&L distributions are the same. The test statistic is $\chi^2$ distributed with 2 degrees of freedom. Significant values at 10%, 5% or 1% levels are marked with one, two or three asterisks, respectively.

#### b. Delta-Gamma Hedge P&L c.d.f.

<table>
<thead>
<tr>
<th></th>
<th>BS</th>
<th>SISC</th>
<th>CEV</th>
<th>Heston (MV)</th>
<th>SABR</th>
<th>SABR (MV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>-</td>
<td>35.212***</td>
<td>1.232</td>
<td>2.327</td>
<td>21.387***</td>
<td>3.507</td>
</tr>
<tr>
<td>SISC</td>
<td>35.183***</td>
<td>-</td>
<td>33.293***</td>
<td>32.409***</td>
<td>3.854</td>
<td>22.970***</td>
</tr>
<tr>
<td>CEV</td>
<td>1.226</td>
<td>33.389***</td>
<td>-</td>
<td>0.742</td>
<td>21.132***</td>
<td>2.211</td>
</tr>
<tr>
<td>Heston (MV)</td>
<td>2.325</td>
<td>32.463***</td>
<td>0.737</td>
<td>-</td>
<td>21.765***</td>
<td>2.696</td>
</tr>
<tr>
<td>SABR (MV)</td>
<td>3.530</td>
<td>23.191***</td>
<td>2.198</td>
<td>2.767</td>
<td>13.132***</td>
<td>-</td>
</tr>
</tbody>
</table>