Global Portfolio Optimization Revisited:
A Least Discrimination Alternative to Black-Litterman

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Abstract

Global portfolio optimization models rank among the proudest achievements of modern finance theory, but practitioners are still struggling to put them to work. In 1992, Black and Litterman recognized the difficulties portfolio managers have in expanding their personal views about some expected asset returns into full probabilistic forecasts about all asset returns and developed a method to facilitate this task. We propose a more general method based on a least discrimination (LD) principle. It produces a probabilistic forecast that is true to personal views but is otherwise as close as possible to a chosen reference forecast. For this purpose we expand the concept of optimal portfolio to include non-linear pay-offs and derive an economic measure of distance – a generalized relative entropy distance – between probabilistic forecasts. The LD method produces optimal portfolios matching any views, including views on volatility and correlation as well as expected returns, and containing option-like pay-offs, if allowed. It also justifies a simple linear interpolation between reference and personal forecasts, should a compromise need be reached.

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I QUANTITATIVE MODELS FOR GLOBAL PORTFOLIO OPTIMIZATION

Academics and practitioners generally agree that quantitative asset allocation models – based on the pioneering work of Markowitz more than fifty years ago – have not played the important role they should in global portfolio management, largely because the supposedly optimal resulting allocations are often too extreme, oversensitive to small input changes and possibly biased (Michaud [1989], Broadie [1993]). Various techniques have been developed over the years to alleviate these problems. They encompass robust estimation techniques of model parameters (see Jorion [1986]), the use of allocation constraints to improve stability and realism (Jagannathan and Ma [2003]) and, more recently, conservative, max-min type, robust optimization techniques (Fabozzi et al [2007]), to name the main strands. Most of these techniques are designed to cope, directly or indirectly, with statistical estimation errors. But all information does not come from statistics and Black and Litterman [1992] raised the more general issue of assisting portfolio managers in making probabilistic forecasts faithfully reflecting their personal views, wherever these views may come from.

Fifteen years on, we observe the Black and Litterman (BL) approach has been much discussed in academic circles but has had only moderate success with professional portfolio managers. One reason may that the BL approach is based on ad hoc assumptions and requires parameters that are not easy to assess. Many subsequent authors tried to explain the intuition behind the BL approach, but we think the approach itself needs to be rebuilt on a firmer rationale. We develop here a more general approach based on a principle of least discrimination.

Formal quantitative decision models are successfully used in many fields. One would not think of running an oil refinery, guiding a spacecraft to a distant planet or pricing an exotic option without such models. Complexity, the need to combine various sources of expertise, sometimes the need to balance the conflicting interests of several concerned parties, justify the use of formal models; modern information technology facilitate their implementation. And yet we would not use formal models to choose, say, where to live and whom to live with – these are not trivial matters but they are usually left to unaided intuition. The optimisation of investment portfolios sits somewhere between these two extremes; one could say that it remains an art as much as a science. On one hand, there is a wealth of relevant information to be processed, there are well-developed financial theories and models to do so, and computers are up to the task of carrying out the most sophisticated analyses. On the other hand, both the specification of investors’ objectives and their forecast of future returns remain ultimately vague and subjective, hence the superficial impression that formal quantitative models for portfolio optimization may not be of much help. But dealing with uncertainty is no excuse for fuzzy reasoning; whether knowledge is vague or precise, formal models help bring coherence in thoughts.
A major challenge in using formal quantitative portfolio optimization models is in the translation of subjective, often qualitative views into sensible quantitative inputs, specifically, in the translation of personal views on some asset returns into probabilistic forecasts for all asset returns. In this paper, we define a personal opinion, or view, as a statement about any characteristic of some asset returns at the investment horizon; it could be a statement about an expected value, volatility, correlation, or some combination thereof. We call probabilistic forecast, on the other hand, a joint probability distribution for all asset returns at the investment horizon. We do not discuss here how portfolio managers form their opinions (i.e., the sources of information and the arguments they use), but focus on the way they express their opinions and on how to interpret what is left unsaid. Indeed, what is left unsaid is supposedly in tacit agreement with some agreed forecast. BL suggest that a portfolio manager should know what asset allocation she would pick by default before doing her research – call this her ‘reference’ or ‘market’ portfolio. It may be the current portfolio she manages, a benchmark against which her performance is measured, or any other portfolio she thinks would best satisfy her investors and fulfil her mandate. According to Markowitz’s one-period, unconstrained portfolio mean-variance analysis, and given a covariance matrix of returns and a statement of investors’ risk attitude, there is a unique set of expected returns for which the reference portfolio is optimal. It is this combination of expected returns and covariances together with an assumption of normality that we shall call the ‘reference’ forecast, or ‘market’ forecast. The closer a personal forecast is to the market forecast, the closer the matching optimal portfolio should be to the market portfolio.

We formalize this approach by deriving a relevant measure of distance between two probability forecasts. Discrepancies between a personal and a market forecast can be exploited by adding an active allocation to the market portfolio, which leads to an expectation of outperforming the market portfolio. We take this expectation, defined as an increase in certainty equivalent, as a distance measure between forecasts and introduce the following principle of least discrimination (LD): the expectation of outperforming the market portfolio should be kept to a minimum, subject to matching the personal views of the portfolio manager. Indeed, that expectation would be illusory if it arose inadvertently rather than as a necessary consequence of consciously held views. The LD principle can be implemented through an iterative process starting with the market forecast and going through a sequence of least discriminatory forecasts taking into account successive refinements of the portfolio manager’s views until she agrees with the last least discriminatory forecast presented to her.

When a portfolio manager agrees with all market implied covariances and has personal views only about some expected returns, we find that the optimal active portfolio consists only of positions, long or short, in the corresponding assets and the risk free asset. But if the portfolio manager has personal
views about volatilities and correlations or is willing to put into question the market forecast of
covariances, the optimal active portfolio is a non-linear pay-off in the asset returns, i.e. an option-like
pay-off. Such pay-off may be constructed by combining listed options or, more generally, by entering
into OTC contracts, or by dynamically investing in the underlying assets.

The optimal active portfolio produced by the LD method may still appear surprisingly large or
unrealistic in some way. As with any formal decision model, the results should never be accepted until
the decision maker is satisfied, on balance, with the complete analysis, that is, with the inputs, the
choice of objective and the logical structure of the model, as well as with the consequences. Laying out
assessments and assumptions to critical examination is what formal models are for. We are therefore
averse to suggesting a recipe for bringing the optimal portfolio closer to the market portfolio. But if, in
the end, the portfolio manager is resigned to mitigate her views in order to achieve such proximity, we
show that a simple weighted average between her personal forecast and the market forecast is a logical
compromise. In the absence of constraints, this weighted forecast leads to scaling down the active
portfolio. If the allocation is constrained, the weighted forecast should be fed into an appropriate
optimizer.

In Section II and Appendix A we analyse the weaknesses of the BL approach and suggest a less
problematic limiting case with a more intuitive parameterization. In Section III and Appendices B and
C we demonstrate the equivalence between a generalized relative entropy (GRE) distance between two
probability distributions and the increase in certainty equivalent that can be achieved with an optimal
active portfolio. We show how to minimise this distance to generate least discriminatory forecasts. A
variety of views and the corresponding optimal allocations, both linear and non-linear, are analysed. In
Section IV and Appendix D we derive weighted forecasts between market and personal forecasts. We
summarise our findings and suggest extensions in Section V.

II A REVIEW OF THE BLACK-LITTERMAN APPROACH

Few people have been trained to translate their knowledge into probabilistic statements; yet, without
training, one may unwittingly mislead or be misunderstood. Expressing imperfect knowledge about an
uncertain quantity in the form of a probability distribution is not a trivial task. Classical statisticians
limit themselves to describing the likelihood of empirical evidence given some hypothesis. To obtain a
posterior probability distribution of the uncertain quantity one needs to start with prior probabilities.
But studies in cognitive sciences have shown that we are all susceptible to numerous prior probabilities.

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general background information is an even greater challenge involving the assessment of conditional probabilities. The BL approach is an attempt to guide the task of combining general market information about asset returns with personal views.

The BL approach was introduced originally in 1990 and refined in 1992. It has been discussed extensively. It has been red-derived, extended and explained by Satchell and Scowcroft [1997], He and Litterman [1999], Drobetz [2001] and Idzorek [2004], among others. Examples of applications have been given by Fabozzi et al [2006] and Jones et al [2007]. But, whereas these authors attempt, as they say, to ‘demystify’, provide ‘intuition’, ‘put to work’ or give a ‘guide’ for using the BL approach, we find that the BL approach relies on several ad hoc assumptions that call for a critical review. We develop here a limiting case of BL with more intuitive parameters, but it still lacks a fundamental rationale. A mathematical description of BL and of this limiting case are given in Appendix A.

**Critique of the Uncertainty Assumptions Underpinning BL**

The BL method rests on two ingenious assumptions. First, BL assume that expected returns are uncertain and interpret the market forecast and personal views as two independent and complementary estimates of expected returns, as if they were readings from two imperfect but independent measuring instruments. Consequently, they calculate a posterior distribution of expected returns by taking the product (re-normalised) of the probability densities of expected returns from the two information sources. The relative degree of accuracy attributed to the market forecast compared to the personal views determines how close the posterior distribution of expected returns is to the market forecast.

Second, they make distributional assumptions about the uncertainties attributed to the two information sources. Specifically, they assume that: (i) the covariance matrix of market expected returns is a scaled down version of the covariance matrix of returns, and (ii) the covariance matrix of personal views about expected returns is diagonal, i.e., uncertainties among personal views are uncorrelated. The choice of covariance matrices determines how views on some expected returns affect other expected returns. For example, if the portfolio manager opines that the expected return of an asset should be higher than according to the market forecast, ipso facto, all positively correlated expected returns are raised and negatively correlated expected returns are reduced. It seems therefore appropriate to question the basis for BL’s distributional assumptions as well as the means of assessing the uncertainties attributed to the two information sources.

How are we meant indeed to quantify uncertainties in opinions about expected returns? We would argue that, no matter how uncertain one may be about some future return, the subjective probability
distribution describing this uncertainty has but one expected value and one would be at a loss – as indeed many portfolio managers are – to express an uncertainty about this expected value (or any other characteristic of the probabilistic forecast of returns, for that matter). This not an argument for single point estimates. On the contrary, we recognize that probability distributions are often specified with a few parameters, for example, an expected value and a variance, about which we have limited information. In such cases, relying on point estimates of parameters would be wrong. We need to integrate the distribution of returns conditional on the value of the parameters over the full distributions of these parameters in order to obtain an unconditional distribution of returns. But asset allocation decisions depend only on the unconditional distribution of returns, it is therefore both inoperational and meaningless to discuss uncertainties in the expected returns of the unconditional distribution. In the case at hand, there is not even the suggestion of a statistical model for arriving at personal views about expected returns; therefore we find no basis for assessing uncertainties about expected returns, and for assuming that such uncertainties would be uncorrelated.

We would argue that uncertainty in the expected value of the market forecast of returns is also irrelevant for the same reason: investment decisions are based on the unconditional distribution of returns which has but one expected value. However, since we use inference models yielding estimates of expected returns, we may be curious to analyse the size and shape of the estimation errors. What inference models are available? There is no information on expected returns in futures and option prices (except when arbitrage between spot and forward prices is impossible, restricted or very expensive). Only vague information can be obtained from time series analysis of historical returns. So, BL rely on the expected returns implied by the choice of a reference portfolio; the logic for deriving an optimal asset allocation from a forecast of asset returns is used in reverse to retrieve a forecast consistent with an optimal asset allocation. According to mean-variance analysis, the relationship between expected returns, $\mu$, the covariance matrix of returns, $\Sigma$, the risk aversion coefficient of the investor, $\gamma$, and the asset weights in the reference portfolio, $\omega$, is simply

$$\mu = \gamma \Sigma \omega$$

The weights are those defining the market portfolio. The covariance matrix of returns can be estimated from historical data and current option prices with a fair degree of accuracy. The coefficient of risk aversion is the least well-known quantity in (1) and is a major source of uncertainty in any estimate of market implied expected returns. Yet, this uncertainty is not critical if the reference portfolio is limited to a universe of risky assets, leaving to the investor the choice of leverage with a risk-free asset, because the coefficient of risk aversion acts as a scaling factor that does not affect relative allocations to risky assets. So, to determine the optimal mix of risky assets, it is sufficient to express all expected returns –
market and personal – in relative terms only. With relationship (1), neither the uncertainty about the coefficient of risk aversion nor the uncertainty about the covariance matrix of returns would justify a multivariate normal distribution for $\mathbf{\mu}$ with a covariance matrix proportional to $\Sigma$.

Finally, BL’s assumption of independence between the two sources of information is dubious. Analysts whose views are sought after are careful students of the markets; they can hardly be expected to provide independent information. Several implausible consequences follow. For example, the posterior forecast of expected returns is more precise than the more precise of the two information sources; the more opinions are gathered, no matter how much they may differ, the more precise the posterior forecast becomes; the posterior distribution is different whether one tacitly does not disagree with a market view or one expressly approves of it.

Instead of relying on this independence assumption, we argue that there is no automatic way to combine information from various sources to form a posterior distribution: it all depends on the degree of credibility attributed to each source and to what extent each source provides independent and complementary information. For example, a portfolio manager may rely on forecasts from two analysts. If they agree, the portfolio manager may adopt their common view. She might even feel slightly more confident if the two analysts arrive at the same conclusion using different arguments – a form of complementarity rather than repetition of information. If the two analysts disagree, the portfolio manager may choose to rely on a mixture of the two forecasts, each given relative credibility weights. So, combining probabilistic forecasts from different sources is ultimately a matter of judgment.

Many techniques (Delphi and others) have been used to combine expert judgments. Depending on circumstances, the posterior distribution is more or less precise than any of the sources.

**BL WITH CREDIBILITY WEIGHTS INSTEAD OF UNCERTAINTY ASSUMPTIONS**

Fortunately, BL’s assumptions about uncertainties in expected returns and how to combine them may not be critical because they are only used as devices for combining market and personal opinions. We show in Appendix A that there is a simpler alternative that does not require these assumptions and therefore avoids some of the undesirable consequences. When uncertainties about expected returns both from market forecasts and personal views are shrunk by the same factor, the posterior expected returns remain unchanged but their distribution is shrunk until it becomes singular when the shrinkage factor becomes nil. We call this limit the BL singular model (BLS); the only required inputs are credibility weights from 0% to 100% for each personal view compared to the corresponding market implied view. Furthermore, in the case where the personal credibility weights are 100% for some
views, the posterior return distributions produced by BL and BLS converge towards the same
distribution conditional on the views held for certain; we call this case the BL conditional (BLC) model.
We illustrate the basic features of the BLC model with a two risky asset portfolio. In the next section,
we use the same portfolio and personal views to illustrate the LD model. We take the volatilities of
risky assets A and B to be 10% and 20% respectively with a correlation of 0.625% (we deliberately
choose these parameters so as to create an unstable optimal portfolio: the two assets have similar
Sharpe ratios and their relatively high correlation reduces the benefits of diversification, thus the
optimal portfolio should be sensitive to small changes in expected returns and volatilities). We assume
that the market portfolio is fully invested half in asset A and half in asset B and that the risk aversion
coefficient of investors is equal to 4. We deduce from (1) that the market forecasts of expected returns
are $\pi_A = 4.5\%$ and $\pi_B = 10.5\%$. The market expected returns and corresponding allocation are
represented as scenario 'M' in Exhibit 1.

Note that if the portfolio manager were to use the BL approach and assign an uncertainty to the market
implied expected returns of, say, 30% of the return volatilities, that is, if she were to choose a scaling
factor $\tau = 0.09$ for the covariance matrix of market expected returns, then the variances of market
returns would be increased by a factor of 1.09; the revised yearly standard deviations would be 10.44%
for A and 20.88% for B. To be consistent, the market implied expected returns should also be revised
upwards by a factor of 1.09 to $\pi_A = 4.95\%$ and $\pi_B = 11.45\%$. The BLS model eliminates these
undesirable adjustments.

Suppose now that the portfolio manager accepts the market covariances for A and B but believes that
the expected returns for A should be slightly higher, namely, $p_A = 5.5\%$ instead of 4.5% according to
the market. She might choose to express her opinion in many different ways, for example, as one of the
following scenarios:

- **P1**: $p_A = 5.5\%$ and $p_B = 10.5\%$ as per market implied forecast
- **P2**: $p_A = 5.5\%$, nothing said about $p_B$
- **P3**: $p_B - p_A = 5\%$
- **P4**: $11p_B - 21p_A = 0\%$

These four statements are not contradictory; superficially, they might even appear similar, but they are
not as we shall see. P1 fixes both $p_A$ and $p_B$ and therefore specifies completely the personal forecast,
there is no room left for any adjustment; P2 fixes $p_A$ but leaves $p_B$ free to be revised away from the
market forecast; P3 and P4 are the least specific statements, fixing neither $p_A$ nor $p_B$, but stating only
some linear relationship between the two. P3 states an absolute difference between the two, whereas P4 states a relative difference. Note that statement P4 should suffice for determining the optimal mix of risky assets.

Applying the BLC model, i.e., with 100% credibility attributed to the personal views, we observe large differences in posterior expected returns and optimal allocations according to the choice of scenario. The results are shown in Exhibit 1. Posterior expected returns for \((p_A, p_B)\) are as high as \((5.5\%, 11.75\%)\) with P2 and as low as \((4.5\%, 9.25\%)\) with P4. Optimal allocations range from 60% to 91% for A, from 37.2% to 50% for B, and from neutrality to 25% borrowing at the risk free rate.

In summary, we have shown that the BLC model achieves the results intended with the BL model but without some of its complexities (assessment of uncertainties about expected returns) and drawbacks (increase in the total variance of returns). However, the posterior expected returns it produces (and therefore the corresponding optimal allocations) seem very sensitive to apparently small variations in the way opinions are expressed. We are left to ponder whether such high sensitivity is simply the consequence of some unwarranted assumptions in the BL method.

III LEAST DISCRIMINATORY PERSONAL FORECAST AND OPTIMAL ACTIVE PORTFOLIO

A simple yet powerful principle in finance is that of no arbitrage, which we restate as follows: based on market information, there are no financial strategies that can lead to gains but no losses. Indeed if there were such strategies, investors, regardless of their risk attitudes, would immediately adopt them and thus weigh on market prices, bringing them back to a no arbitrage equilibrium. But what if an individual does not fully agree with the market forecast? What expectations can he reasonably hold? We propose the following least discrimination (LD) principle: among all forecasts satisfying a set of personal views, one should select the forecast that yields the minimum increase in certainty equivalent when optimally acted upon. Any other forecast would lead to gains in certainty equivalent not supported by personal views and therefore illusory. No arbitrage is commonly described as a ‘no free lunch’ principle, least discrimination could be described as ‘no illusion’ principle.

Strategies exploiting personal views generally lead to uncertain consequences and therefore must be evaluated in terms of risks as well as returns. This is why the LD principle is expressed in terms of certainty equivalent (minimum selling price of a risky opportunity), which reflects the risk attitude of the investor. Certainty equivalents could be assessed intuitively by investors on a case by case basis, but
it is more efficient to encode the risk attitude of an investor in a utility function and then use it to calculate his certainty equivalent for any risky opportunity.

**Optimal Pay-off and Certainty Equivalent**

To evaluate certainty equivalents over a wide range of investment strategies, we first extend the concept of optimal portfolio to that of optimal pay-off. An optimal portfolio is usually understood as an optimal static asset allocation yielding a return that is a linear combination of the component asset returns. More generally, an investment strategy may lead to a non-linear, i.e., option-like, pay-off in the component assets returns. It may be generated by option positions or a dynamic asset allocation strategy.

In Appendix B we show that, given a risk neutral forecast of returns $q(\mathbf{r})$, a personal forecast $p(\mathbf{r})$, and a utility function of wealth $u(x)$, the pay-off maximizing expected utility is such that the marginal utility $u_x(x(\mathbf{r}))$ should be proportional to $q(\mathbf{r})/p(\mathbf{r})$. The optimal pay-off can be calculated for any utility function but is particularly easy to express analytically when choosing an exponential utility function because it is independent of initial wealth. In that case the optimal 'par' pay-off (i.e., pay-off which can be traded at zero cost) is of the form:

$$f_{q,p}(\mathbf{r}) = \left(\frac{1}{\gamma}\right) \left[\ln(p(\mathbf{r})/q(\mathbf{r}))/\gamma + D(q, p)\right]$$

(2)

where $\gamma$ is the coefficient of risk aversion of the investor and $D(q, p)$, a scalar independent of $\mathbf{r}$, is the relative entropy distance between distributions $q(\mathbf{r})$ and $p(\mathbf{r})$ defined as $D(q, p) = E_q[\ln(q(\mathbf{r})/p(\mathbf{r}))]$.

Remarkably, $(1/\gamma)D(q, p)$ is also the increase in certainty equivalent that is achieved by choosing the optimal pay-off rather than investing in the risk free asset. When $q(\mathbf{r})$ and $p(\mathbf{r})$ are multivariate normal distributions with same covariance matrix, formula (2) yields Markowitz’ optimal portfolio allocation and the gain in certainty equivalent is the quadratic form $\frac{1}{2}p'(\gamma \Sigma)^{-1}p$, where $p$ is the vector of expected returns according to the personal forecast $p(\mathbf{r})$.

If the personal forecast $p(\mathbf{r})$ differs from the market forecast $m(\mathbf{r})$, the corresponding optimal pay-off $f_{q,p}(\mathbf{r})$ differs from the market portfolio $f_{q,m}(\mathbf{r})$. Call $f_{q,m,p}(\mathbf{r})$ the difference between these two pay-offs, or optimal active pay-off. We show in Appendix B that

$$f_{q,m,p}(\mathbf{r}) = \left(\frac{1}{\gamma}\right)(\ln[p(\mathbf{r})/m(\mathbf{r})] - E_m[\ln[p(\mathbf{r})/m(\mathbf{r})]])$$

(3)

and that the increase in certainty equivalent from adding this active pay-off to the market portfolio is:

$$(1/\gamma)D(q, m, p) = (1/\gamma) \left(\ln(E_m[p(\mathbf{r})/m(\mathbf{r})]) - E_m[\ln(p(\mathbf{r})/m(\mathbf{r})])\right)$$

(4)
The quantity $D(q, m, p)$ so defined can be interpreted as a generalized relative entropy (GRE) distance between distributions $m(r)$ and $p(r)$ from the $q(r)$ perspective. It is a distance because it is always non-negative and equal to zero if and only if $p(r) = m(r)$ almost everywhere.

We envisage the implementation of the LD principle as an iterative process starting with the market forecast and concluding with a least discriminatory forecast satisfying the portfolio manager. At every step the portfolio manager reviews critically the last forecast presented to her and her opinions are used to produce a new revised forecast in the form of a new least discriminatory distribution (LDD). The process of manipulating LDDs could be likened to that of a sculptor modeling a lump of clay until the right figure emerges.

The construction of an LDD is particularly simple when risk attitude is described by an exponential utility function; then the potential increase in certainty equivalent is given by (4) and closed form solutions to the certainty equivalent minimization problem may be found. In particular, when the risk neutral forecast and the market forecast are multivariate normal distributions and personal views are limited to expected returns and covariances of returns, then the LDD is also a multivariate normal distribution. The optimal active portfolio is a linear form in $r$ when the personal forecast agrees with the market covariance matrix and a quadratic form in $r$ when it does not. Calculations of LDDs are discussed in Appendix C.

**OPTIMIZING ON EXPECTED RETURNS GIVEN MARKET COVARIANCES**

When the portfolio manager adopts the market covariance matrix but expresses different views about some expected returns in the form of linear conditions, we find that the LD principle leads to the same results as the BLC approach. These findings give support to the choice of uncertainty assumptions made by Black & Litterman as implemented in the BLC case. Other choices of error distributions would lead to results incompatible with the LD approach.

The LD method also yields the gain in certainty equivalent for each optimal portfolio relative to the market portfolio (using (C13) in this case). Exhibit 1 (last column) displays gains in certainty equivalent for the four scenarios P1 to P4 used in Section II to illustrate the BLC method. We find that P1, the most specific view, generates the largest gain of 21 bp followed by P4, which is the same as P1 but expressed in relative terms, with a gain of 16 bp. P2, which specifies only the expected return of asset A and lets the model find the conditional expected return on asset B that minimizes the distance to the market view, yields a gain in certainty equivalent of 12.5 bp. And finally P3, which imposes only a condition on the spread between the expected returns of A and B, yields a gain of only 5bp.
An examination of (C13) sheds light on these results; (C13) states that forecasts leading to equal gains in certainty equivalent are along ellipsoids centered at the market expected returns. In our two risky asset example, the ellipse corresponding to a gain of 16 bp is drawn in Exhibit 1. According to view P2, which states only that $p_A = 5.5\%$, the value of $p_B$ that is least discriminatory, i.e., yields the lowest increase in certainty equivalent is $p_B = 11.75\%$ because that is the point of tangency of the line $p_B = 5.5\%$ with the ellipse of allocations 16 bp away from the market (one can verify that the beta of asset B versus asset A is equal to $0.625 \times (0.2)/(0.1) = 1.25$ so that when the expected return on A is increased by 1%, the expected return on B is increased by 1.25%). Scenario P1 is more constraining because it also sets $p_B = 10.5\%$, as a result P1 is more distant from the market view than P2. We also verify that points P3 and P4 are tangency points with ellipses at distances of 5 bp and 16 bp respectively from the market.

The variety of optimal portfolios and gains in certainty equivalent according to scenarios highlights the importance describing opinions in the most faithful way. The portfolio manager should ponder which expression P1 through P4, or perhaps others, best reflect her belief that asset A will perform better than implied by the market. Is it that the expected return on A should be 1% higher than implied by the market with nothing else changed (as in P1), or should she also accept to increase her forecast on asset B because she trusts the positive correlation between asset A and B (as in P2), or should she not put into question the market forecast of volatilities and correlation? Are her views simply about the relative performance of A and B (as in P3 and P4)? If the portfolio manager cares only about the risky asset mix and leaves the leveraging decision to investors, a view expressed in relative terms, as in P4, is sufficient to determine the optimal mix; but does it understate the views of portfolio manager?

**Optimizing on Expected Returns given any Covariance Matrix**

BL rely on the acceptance of market implied covariance matrix and does not extend easily to personal views on volatilities and correlations. We argue that minimizing the GRE distance and therefore minimizing the increase in certainty equivalent that can be obtained by exploiting personal views may be applied to any type of views. Using this approach, we find in particular that expected returns should be adjusted to minimize the impact of views on covariances in such a way that the active portfolio contains only non-linear pay-offs. In other words, the expected returns are adjusted so that the reference portfolio is still the optimal linear portfolio with the new covariance matrix. One may find something intuitively pleasant about this result. Vice versa, covariances may be adjusted to reduce the impact of views on expected returns. These adjustments are explained in Appendix C. Of course, the GRE minimization process is not analytically tractable except in a few simple cases, but widely available
numerical solvers (e.g., the solver available in Excel) can be used without difficulty – at least when the number of parameters is not very large – because the GRE distance is a well behaved objective function.

To gain insight into these adjustments we use two illustrations based on the previously analysed two-risky asset portfolio. In the first illustration we stipulate covariances different from the market implied covariances and use an analytical formula to derive the adjustment to expected returns. In the second illustration we revisit our previously examined scenarios about expected returns and use a solver to find the adjustments to covariances as well as to expected returns that minimize the GRE.

The market implied correlation between assets A and B in our previous example is 62.5%. Suppose now that the portfolio manager wants to revise this figure down to 37.5% for the investment horizon, volatilities remaining as per market forecast. If the portfolio manager accepts to have her forecast of expected returns adjusted because of her view on correlation, expression (C8) in Appendix C gives the solution. The new expected returns in that case are \( p_A = 3.5\% \) and \( p_B = 9.5\% \). They are such that the reference portfolio with 50% allocations to each of A and B is still optimal except for the need to introduce a quadratic active pay-off. The active pay-off is \( f_{q,m,p}(r) = 5.97 \ v_A^2 - 7.37 \ \sigma_{AB} + 1.49 \ v_B^2 - 0.027 \) (from (B9) with coefficients rounded to two decimal places) and is higher when \( r_A \) and \( r_B \) are of opposite signs than when they are of same sign, thus taking advantage of a lower correlation than the market implied correlation. The corresponding gain in certainty equivalent is 1.57% (from (B10)).

If expected returns were maintained at their implied market forecast of \( p_A = 4.5\% \) and \( p_B = 10.5\% \), the optimal active pay-off would contain additional weights of 23.6% in A and 1.8% in B (and therefore a borrowing at the risk free rate of 25.4%). The corresponding gain in certainty equivalent would be 1.70%; but this gain and the additional active weights in both assets are unwarranted unless the portfolio manager insists on holding on to the market forecast of expected returns.

The quadratic active pay-off can be constructed with option positions. The coefficients of the quadratic active pay-off determine the required gammas and therefore the required options. For example, the above pay-off could be approximated with the following 3-month straddles (a long put and a long call at-the-money forward): long 0.29 straddles on A, short 0.55 straddles on B and long 0.86 straddles on the spread (A – B). This calculation is based on approximate gammas, namely, for an ATM forward straddle on asset A, \( (0.8/\sigma_A) \), and for an ATM forward straddle on the spread (A – B),
(0.8/σ_A) with respect to asset A, (0.8/σ_B) with respect to asset B, and -(0.8/σ_{A,B}) for the cross gamma. On a basket (A+B) the cross gamma would be + (0.8/σ_{A+B}).

**OPTIMIZING ON BOTH EXPECTED RETURNS AND COVARIANCES**

We now revisit our previously examined scenarios about expected returns assuming that covariances may be adjusted to minimize the GRE. This is an example of the general situation in which any views can be imposed on some expected returns and covariances and we minimize the GRE with respect to what is left as free parameters. This situation calls for the use of a general solver.

The results for scenarios P1 to P4 are shown in Exhibits 2 and 3. For all scenarios, adjusting covariances leads to further reductions of the GRE and a smaller linear active portfolio, however the active portfolio now contains quadratic terms, i.e., requires option positions. Differences with previous optimal allocations are shown in Exhibit 2 with arrows linking the old allocations to the new allocations for each scenario. Take for example scenario P1 setting both expected returns (p_A = 5.5%, p_B = 10.5%). Without covariance adjustments, the optimal allocation is 91% for A, 37.2% for B and a borrowing of 28.2% leading to a gain in certainty equivalent of 20.5 bp. However, allowing for covariance adjustments, the new optimal allocation consists of 79.3% for A, 39.7% for B and a borrowing of 19%. This is much closer to the 50%/50% market portfolio and implies a gain in certainty equivalent of only 15 bp. However this implies also a small quadratic active pay-off as shown in Exhibit 3, because views on volatilities and correlation have been adjusted by small amounts to 10.36% for A, 19.95% for B and 63.88% for the correlation.

In this instance, the quadratic pay-off contains the product of the return, r_Ar_B, and therefore requires a correlation dependent option, for example a straddle on the spread (A-B). Unfortunately such options are usually traded in not very liquid OTC markets and the bid-ask spreads may be expensive. One can only hope that such products, or specially designed correlation futures and options, will become more available in the future so that investors exercise their views on correlations. In the meantime, if a portfolio manager cannot deal in such products, she will want an optimal portfolio without cross product terms between asset returns. It is a simple matter to add this constraint in the solver; it produces the optimal results shown on the last line of Exhibits 2 and 3 under scenario P10. Scenario P10 leads to an optimal portfolio 82.8% in A and 41.2% in B, therefore not as close to the market portfolio as when correlation is adjusted. Nonetheless, it is not as far from the market portfolio as when volatilities are not adjusted. Because the volatility of asset A is adjusted upwards from 10% to 10.19% and the volatility of B is adjusted downwards from 20% to 19.63%, the optimal portfolio
includes a small short volatility position on A and a small long volatility position on B (with three
months at-the-money forward straddles, and using the same approximate gammas as before, the
positions would be short 0.096 staddles on A and long 0.047 straddles on B.

This example illustrates how tricky the expression of opinions may be and the complex interactions
with the constraints that must be taken into account to arrive at a realistic optimal portfolio. For these
reasons, we would not expect a portfolio manager to be able to state definitely in a single step the views
she wants to implement. It is more likely that the final expression of views and constraints should
emerge from an iterative process using the feedback from successive optimization results. The key is
the choice of objective function for the optimization; the optimisation process itself is simple enough
with modern solvers and can handle a wide variety of views.

IV BLENDING MARKET AND PERSONAL FORECASTS

As we mentioned at the outset, there are various reasons why a supposedly optimal portfolio produced
by a formal asset allocation model may appear unrealistic or unattractive to a portfolio manager: the
optimal active portfolio may be appear too large or it may be highly sensitive to small changes in views.
The latter is a problem if personal views are expected to evolve over time and transaction costs are too
high or market liquidity too low to allow for rapid changes in asset allocations. The issue of designing
optimal dynamic allocation strategies in such circumstances has been addressed by Davis et al [1990]
among others. But if a portfolio manager is simply unhappy that her LDD forecast produces an
optimal portfolio too radically different from the benchmark she chose, she should re-examine her
forecast and the reasons why she chose that particular benchmark to try to reconcile her views with
their consequences. As a last resort only, should she rely on a mechanistic approach to reach a
compromise between her forecast p(r) and the market forecast m(r). The question then becomes: what
is the best family of blended forecasts b(r) bridging the gap between market and personal forecasts?

Based on the LD principle and GRE as the appropriate distance between probability distributions, it is
reasonable to consider that among all blended forecasts a given distance away from the market forecast,
a portfolio manager should select the one that is closest to her personal forecast. In Appendix D, we
derive the unique solution to this problem when forecasts are represented by multivariate normal
distributions. We find that the best blended forecasts are simple linear interpolations between market
and personal forecasts. In other words, all expected returns and covariances from market and personal
forecasts are combined with the same weights (note in passing that correlations are not combined with
the same weights unless covariances in the personal forecast are proportional to those of the market
forecast).
The BL model is not compatible with this linear interpolation. The blended posterior expected returns (formula (A3)) are not obtained by interpolating expected returns between market and personal views held with certainty (formula (A6)) barring exceptional circumstances such as when personal views are only about uncorrelated returns. To illustrate, consider scenario P5 in Exhibit 1. It is designed as giving equal credibility to scenario P1 and the market forecast. The BLS approach is specially designed to handle such blended views. We might well expect that the equal credibility weights would lead to the average posterior forecast (5%, 10.5%) – average between the market (4.5%, 10.5%) and scenario P1 (5.5%, 10.5%) – but, surprisingly, BLS yields the posterior forecast (4.95%, 10.85%). In contrast, the LD method justifies the linear interpolation (5%, 10.5%) shown as P6 in Exhibit 1. The optimal allocations to risky assets A, B under P5 are 63.8%, 47.8% with borrowing of 11.7%, whereas under P6 they are 70.5%, 43.6% with a borrowing of 14.1%. Similarly surprising results would be obtained with BLS applied to the blending of other scenarios with the market forecast. The two methods lead to significantly different results both in terms of blended forecasts and in terms of corresponding optimal allocations. An empirical application of the linear interpolation technique can be found in Pezier and White [2006].

V SUMMARY AND CONCLUSIONS

Black and Litterman examined how partial views may be best expanded into a full probabilistic forecast. Their solution— to assume that what is left unsaid must be in close agreement with some shared market forecast – is intuitive. But their method, though ingenious, is unnecessarily complex, limited in scope and not founded on a clear principle. Also, they put great emphasis on limiting the credibility attached to personal views in order to bring the optimal portfolio close to the market portfolio. In contrast, our primary objective is to produce a probabilistic forecast that remains true to the views expressed by a portfolio manager, whatever they may be, whilst remaining as close as possible to the market forecast on other matters. To this end we have derived a distance measure between forecasts based on the gains that can be expected from holding views that differ from the market forecast and have proposed the following principle of least discrimination: one should not expect gains not supported by expressed opinions. We have also found that the blending of personal and market views should take the form of a simple linear interpolation, but regard this procedure as a last resort for those who do not have the discipline or time for refining their own opinions.

Minimizing our distance measure is a straightforward process. In some simple instances, closed form analytical solutions are available; they shed light on the reasons why some parameters are adjusted in the minimization process. For example, views on covariances call for an adjustment of expected returns.
so that the optimal linear portfolio remains as in the market portfolio. But to be fully exploited, such views require the addition of non-linear, option like positions. Our method indicates the optimal mix of straight positions and options positions and should encourage portfolio managers to make more systematic use of derivative products. There are liquid markets in a large number of single asset options, but correlation sensitive derivatives, e.g., spread options, basket options, currency protected options, correlation futures and options, are still relatively underdeveloped or non-existent. Understanding better when to use such products should contribute to their growth.

Closed form analytical solutions will generally not be available to cope with the large number of assets, the potential multiplicity and complexity of personal views and the operational constraints a portfolio manager needs to take into account. However, our objective function can be easily calculated and used in a general solver; so, portfolio managers should find it easy to explore the consequences of their views and to revise them accordingly until they reach satisfactory combination of forecast and optimal portfolio. The next stage is to examine how this methodology can be expanded into optimal dynamic asset allocation strategies taking into account transaction costs and evolving views.
APPENDIX A – MATHEMATICS OF THE BL MODEL AND TWO SPECIAL CASES

BL consider expected returns $\mu$ on risky assets as uncertain quantities about which the market and personal views provide some information. From a reference portfolio with asset allocation $\omega$, a market estimate of the covariance matrix of returns $\Sigma$ and an investor risk aversion coefficient $\gamma$, one can deduce, using mean-variance analysis, the implied expected returns

$$\pi = \gamma \Sigma \omega$$  \hspace{1cm} (A1)

BL take this estimate to be the expected value of a multivariate normal distribution $\mu \sim N(\pi, \tau \Sigma)$ where $\tau$ is a scaling factor generally chosen to be smaller than 1.

Personal views about expected returns are expressed by a set of linear equations:

$$P\mu = q + \varepsilon$$ \hspace{1cm} (A2)

with normally distributed independent ‘error’ terms $\varepsilon$, i.e., $\varepsilon \sim N(0, \Omega)$, with $\Omega$ a diagonal covariance matrix. The error terms are interpreted as uncertainties in personal opinions about expected values. Consequently, personal views translate into a multivariate normal distribution $P\mu \sim N(q, \Omega)$.

BL then update the market forecast with the personal views. Bayes’ equation states that $p(\mu | q, \pi)$ is equal to the product $p(\mu | \pi) \cdot p(q | \mu, \pi)$ normalized. The first term in this product is the distribution $\mu \sim N(\pi, \tau \Sigma)$; the second term cannot be defined without making further assumptions. BL assume that personal views are independent of the market forecast, that is, $p(q | \mu, \pi) = p(q | \mu)$ and that the latest distribution is $q \sim N(P\mu, \Omega)$. With these assumptions and no personal views held with certainty, i.e., $\Omega$ invertible, the parameters of the posterior distribution of expected returns $\mu \sim N(p, M)$ are

$$p = [\Sigma^{-1} + P'\tau \Omega^{-1}P]^{-1}[\Sigma^{-1} \pi + P'\tau \Omega^{-1}q]$$ \hspace{1cm} (A3)

$$M = \tau[\Sigma^{-1} + P'\tau \Omega^{-1}P]^{-1}$$ \hspace{1cm} (A4)

The posterior distribution of asset returns becomes $r \sim N(p, \Sigma + M)$. In all logic, since $M = \tau \Sigma$ in the absence of personal views, $\Sigma$ in (A1) should be replaced by $(1 + \tau)\Sigma$, $\pi$ in (A3) should be replaced by $(1 + \tau)\pi$ and $p$ should be recalculated accordingly.

But we can leave the covariance matrix of returns unchanged whilst preserving the main effects of BL by considering the limiting case where the posterior distributions of expected returns becomes singular. We note that in (A3) $p$ does not depend on $\tau$ and $\Omega^{-1}$ separately but only on their product $\tau \Omega^{-1}$ and that in (A4) the covariance matrix $M$ tends towards zero with $\tau$. Therefore, if we take the limit when $\tau$
and $\Omega$ are reduced to zero whilst keeping $\tau \Omega^{-1}$ constant, $M$ vanishes, $p$ is unchanged and so is the covariance matrix of returns. We call this limiting case the Black-Litterman Singular (BLS) model and suggest an intuitive parameterization. The diagonal elements of $\Omega$ are the variances assigned to personal views, the diagonal elements of $P\tau \Sigma P'$ are the variances for the corresponding market views. Personal and market views are weighted in the BL approach according to the reciprocals of their variances. This weighting scheme can be restated by defining a diagonal matrix $C$ of credibility weights for personal views, each weight being assigned a value in the semi-open interval $[0, 1)$ and setting $\tau \Omega^{-1}$ for use in (A3) and (A4) as follows:

$$\tau \Omega^{-1} = \text{Diag}[(P\tau \Sigma P')^{-1}]C(I - C)^{-1}$$  \hspace{1cm} (A5)

When $P$ and $q$ describe personal views held with certainty, $\Omega$ is not invertible and the posterior distribution of returns becomes the conditional return distribution given $P\mu = q$; the conditional expected returns are:

$$p = \pi + \Sigma P'(P\Sigma P')^{-1}[q - P\pi]$$  \hspace{1cm} (A6)

We call this special case the Black-Litterman Conditional (BLC) model.

**APPENDIX B – GENERALIZED RELATIVE ENTROPY AS A GAIN IN CERTAINTY EQUIVALENT**

Consider an investment universe with a risk free asset and $n$ risky assets with excess returns $r$ over a certain investment horizon. Any asset and, more generally, any pay-off defined on the returns (derivative product) can be bought or sold at fair price under a risk neutral pdf $q(r)$. An investor entertains a personal forecast $p(r)$. We seek the pay-off $f_p(r)$ that maximises the expected utility of the investor’s wealth at the investment horizon. In mathematical terms, we seek to maximize over the choice of pay-off $f_p(r)$ the expected utility:

$$E_p[u(f_p(r)) - E_q[f_p(r)] + w_0]$$  \hspace{1cm} (B1)

where $E_q[f_p(r)]$ is the cost of the pay-off $f_p(r)$, $w_0$ the initial wealth of the investor and $u(.)$ the investor’s utility.

The optimum pay-off is obtained by setting to zero the first order derivative of the expected utility with respect to $f$, that is, (with $u_x$ denoting the first derivative of $u(x)$ with respect to $x = f_p(r) - E_q[f_p(r)] + w_0$), for any $r$:

$$p(r).u_x(x(r)) - q(r).E_p[u_x] = 0$$  \hspace{1cm} (B2)

This obtains for $u_x(x(r))$ proportional to $q(r)/p(r)$.
The optimal pay-off is particularly easy to calculate with an exponential utility function because it does not depend on the level of initial wealth, which we can therefore set to zero without loss of generality. With \( u(x) = -\exp(-\gamma x) \), it follows that \( u_x = \gamma \exp(-\gamma x) \) and (B2) yields the optimal net pay-off
\[
 f_{q,p}(r) = \frac{1}{\gamma} \ln\left(\frac{p(r)}{q(r)}\right) + C(f_{q,p}|q, p) \quad (B3)
\]
where \( f_{q,p}(r) = f_q(r) - E_q[f_p(r)] \) is the optimal net pay-off and \( C(f_{q,p}|q, p) \) is a scalar independent of \( r \). Substituting (B3) into (B1) with the exponential utility function gives the maximum expected utility
\[
 E_p[u(f_{q,p}(r))] = -\exp(-\gamma C(f_{q,p}|q, p))
\]
Therefore, \( C(f_{q,p}|q, p) \) is the certainty equivalent of the optimal net pay-off \( f_{q,p}(r) \). It is obtained as a function of \( p(r) \) and \( q(r) \) by taking the expected value of both sides of (B3) under the risk neutral distribution \( q(r) \). The result is:
\[
 C(f_{q,p}|q, p) = \frac{1}{\gamma} E_q[\ln\left(\frac{q(r)}{p(r)}\right)] = \frac{1}{\gamma} D(q, p) \quad (B4)
\]
where we recognize the relative entropy distance \( D(q, p) \) between pdfs \( q(r) \) and \( p(r) \).

We now evaluate the active portfolio an investor should add to the market portfolio when he entertains a personal forecast \( p(r) \) different from the market implied forecast \( m(r) \). Using (B3) and (B4), we find by difference the active net pay-off:
\[
 f_{q,m,p}(r) := f_{q,p}(r) - f_{q,m}(r) = \frac{1}{\gamma} \left( \ln\left(\frac{p(r)}{m(r)}\right) + D(q, p) - D(q, m) \right) \quad (B5)
\]
The gain in certainty equivalent from adding this active net pay-off is the difference between the certainty equivalent \( C(f_{q,p}|q, p) \) of receiving the net pay-off \( f_{q,p}(r) \) and the certainty equivalent \( C(f_{q,m}|q, p) \) of receiving the pay-off \( f_{q,m}(r) \) for an investor forecasting \( p(r) \). We evaluate the latest as follows using again (B3) and (B4):
\[
 -\exp(-\gamma C(f_{q,m}|q, p)) = E_p[-\exp(-\ln(m(r)/q(r)) - D(q, m))]
 = -\exp[\ln(E_q[p(r)/m(r)]] - D(q, m))]
\]
and therefore
\[
 C(f_{q,m}|q, p) = \frac{1}{\gamma} \left[ D(q, m) - \ln(E_q[p(r)/m(r)]) \right]
\]
Thus, the gain in certainty equivalent from adding the optimal active portfolio is:
\[
 C(f_{q,p}|q, p) - C(f_{q,m}|q, p) = \frac{1}{\gamma} \left[ \ln(E_q[p(r)/m(r)]) - E_q[\ln(p(r)/m(r))] \right] := \frac{1}{\gamma} D(q, m, p) \quad (B6)
\]
\( D(q, m, p) \) can be interpreted as a generalized relative entropy distance (GRE) between distributions \( m \) and \( p \), based on a third distribution \( q \). One can verify that \( D(q, m, p) \) is always non-negative and is equal to zero only when the distributions \( m \) and \( p \) are the same, except possibly on the null set of \( q \) (from Jensen’s inequality). But \( D(q, m, p) \) is not a metric as, in general, it is not symmetrical in \( m \) and \( p \) and does not respect the triangular inequality. Note also that \( D(m, m, p) = D(m, p) \).
When distributions $q$, $m$, and $p$ are maximum entropy distributions satisfying conditions on first and second order moments, that is, when they are multivariate normal distributions (Tribus [1969]), the optimal active portfolio $f_{q,m,p}(r)$ and the distance $D(q, m, p)$ can be expressed simply as a function of the parameters of these distributions. So we consider the case where $q$ is a risk neutral distribution $N(0, \Sigma \Sigma \Sigma \Sigma)$ and $m$ and $p$ are represented by $N(\pi \pi \pi \pi, \Sigma \Sigma \Sigma \Sigma)$ and $N(p, S)$, respectively. In this case, according to (B3) and (B4), the market portfolio is the linear pay-off:

$$f_{q,m}(r) = \pi'(\gamma \Sigma \Sigma \Sigma \Sigma)^{-1}r$$  \hspace{1cm} (B7)

It can also be written as $f_{q,m}(r) = \omega' r$ with $\omega = (\gamma \Sigma \Sigma \Sigma \Sigma)^{-1} \pi$. It is the optimal static asset allocation obtained by traditional mean-variance analysis. The corresponding certainty equivalent is

$$\frac{1}{\gamma}D(q, m) = \frac{1}{2}\pi'(\gamma \Sigma \Sigma \Sigma \Sigma)^{-1} \pi$$  \hspace{1cm} (B8)

which is always non-negative ($\Sigma \Sigma \Sigma \Sigma$ being positive definite) and equal to zero only if $\pi = 0$.

The optimal active pay-off and the corresponding increase in certainty equivalent are:

$$f_{q,m,p}(r) = \frac{1}{\gamma} (p - S \Sigma \Sigma \Sigma \Sigma \pi)' S^{-1}r - (1/\gamma^2) \left[ r'(S^{-1} - \Sigma^{-1})r + n - \text{Tr}(\Sigma S^{-1}) \right]$$  \hspace{1cm} (B9)

$$\frac{1}{\gamma} D(q, m, p) = \frac{1}{2} (p - S \Sigma \Sigma \Sigma \Sigma \pi)' (\gamma \Sigma \Sigma \Sigma \Sigma)^{-1} (p - S \Sigma \Sigma \Sigma \Sigma \pi) + (1/2\gamma) [\text{Tr}(\Sigma S^{-1}) - n - \ln(|\Sigma S^{-1}|)]$$  \hspace{1cm} (B10)

where $\text{Tr}(\cdot)$ denotes a trace and $|\cdot|$ a determinant. The increase in certainty equivalent is the sum of two non negative terms. The first term is nil if and only if $p = S \Sigma \Sigma \Sigma \Sigma \pi$ when $S$ is positive definite. The second is nil if and only if $S = \Sigma^{-1}$ (the trace of a matrix is the sum of its eigenvalue and the determinant is the product of its eigenvalues and we know that $(x - 1) \geq \ln(x)$ with equality iff $x = 1$).

We observe from (B9) that the term linear in asset returns (the traditional optimal static asset allocation) is of the form $\omega' r$ with allocations $\omega = (\gamma \Sigma \Sigma \Sigma \Sigma)^{-1}(p - S \Sigma \Sigma \Sigma \Sigma \pi)$. The optimal active pay-off contains only quadratic terms in the returns when $p = S \Sigma \Sigma \Sigma \Sigma \pi$. Indeed since $(\gamma \Sigma \Sigma \Sigma \Sigma)^{-1} \pi$ is the market portfolio, $p$ so calculated is the expected return corresponding to the market portfolio under covariance matrix $S$. We also observe from (B10) that the locus of expected returns $p$ in distributions $N(p, S)$ that are equidistant from the market forecast $N(\pi, \Sigma \Sigma \Sigma \Sigma)$ is an ellipsoid centered on $S \Sigma \Sigma \Sigma \Sigma \pi$. This observation gives a geometric interpretation to the construction of LDDs in Appendix C.

In the special case where $S = \Sigma$, the terms in square brackets in (B9) and (B10) are nil and these two expressions reduce to:

$$f_{q,m,p}(r) = (p - \pi)' (\gamma \Sigma)^{-1}r$$  \hspace{1cm} (B11)

$$\frac{1}{\gamma} D(q, m, p) = \frac{1}{2} (p - \pi)' (\gamma \Sigma)^{-1} (p - \pi)$$  \hspace{1cm} (B12)
To illustrate, consider an investor with a risk aversion coefficient $\gamma = 4$ who is given the opportunity to invest in a single risky asset with excess return of 8% and volatility of 20% according to the market forecast. If he agrees with the market, he should, according to (B7), allocate half of his wealth to the risky asset and the other half to the risk free asset; this would, according to (B8), increase his certainty equivalent by 2% compared to investing exclusively in the risk free asset.

But if the same investor changes his mind and now puts the expected return of the risky asset at 6% only, he should, according to (B9), reduce his holding of the risky asset by one eighth; this would increase his certainty equivalent by 0.125%, according to (B10). One verifies that comparing the latter view directly with the risk neutral view leads immediately to an optimal investment of $3/8$ in the risky asset and a gain in certainty equivalent of 1.125%. As is evident from the form of expression (B5), the addition of active pay-offs corresponding to successive changes in views always leads to the optimal pay-off matching the last views. However, successive increases in certainty equivalents do not add up to the increase in certainty equivalent from the initial pay-off to the final pay-off.

**Appendix C – Least Discriminatory Distribution (LDD) and Optimal Allocation**

Consider a market forecast of returns $r$ with pdf $m(r)$ and a set of personal views about some first and second order moments of returns expressed by the following linear equations (the conditioning equations):

$$\mathbf{Pp} = \mathbf{q}$$  \hspace{1cm} (C1)

$$\mathbf{G}.\text{Vech} (\mathbf{S}) = \mathbf{h}$$  \hspace{1cm} (C2)

In these conditioning equations, $\mathbf{P}$ and $\mathbf{G}$ are matrices expressing linear combinations of expected returns and covariances, respectively; $\mathbf{p}$ and $\mathbf{S}$ are the vector of expected return and the covariance matrix of returns in the personal forecast $p(r)$; $\mathbf{q}$ and $\mathbf{h}$ are the column vectors of parameters specifying the views on expected returns and covariances.

We call least discriminatory distribution (LDD) with respect to the market forecast $m(r)$ and the risk neutral distribution $q(r)$ the personal distribution of returns $p(r)$ that satisfies the conditioning equations and minimises the potential increase in certainty equivalent from exploiting optimally the forecast $p(r)$. With exponential utilities, this means minimizing the distance $D(q, m, p)$ defined in (B6) and therefore minimizing the Lagrangian

$$L = D(q, m, p) - \lambda_0 (E_q[1] - 1) - \lambda_1 (\mathbf{Pp} - \mathbf{q}) - \lambda_2 (\mathbf{G}.\text{Vech}(\mathbf{S}) - \mathbf{h})$$  \hspace{1cm} (C3)

where $\lambda_0$ is the Lagrange multiplier corresponding to the normalisation condition for $p(r)$ and $\lambda_1$ and $\lambda_2$ are column vectors of Lagrange multipliers for the conditions on first and second order moments.
We explore this minimization problem in a few simple instances, assuming that the risk neutral forecast $q(r)$ and the market forecast $m(r)$ are multivariate normal distributions $N(0, S)$ and $N(\pi, \Sigma)$ respectively; consequently, the GRE distance is given by (B10).

**C1 – Views on Complete Covariance Matrix and on some Expected Returns**

The minimization of the GRE distance under condition (C1) and $S$ being set leads to minimizing over $p$ the Lagrangian:

$$L = \frac{1}{2}(p - S\Sigma^{-1}\pi)'S^{-1}(p - S\Sigma^{-1}\pi) + \frac{1}{2}\text{Tr}(\Sigma S^{-1}) - n - \ln(|\Sigma S^{-1}|)) - \lambda'_1(p - q) \quad (C4)$$

We set the differentials with respect to $p$ equal to zero:

$$\partial L / \partial p = S^{-1}(p - S\Sigma^{-1}\pi) - P\lambda = 0$$

hence,

$$p = S\Sigma^{-1}\pi + SP\lambda_1$$

Now, setting the differentials with respect to $\lambda_1$ equal to zero (that is, applying condition (C1)) gives

$$\lambda_1 = (PSP')^{-1}(q - P S\Sigma^{-1}\pi) \quad (C5)$$

so that

$$p = S\Sigma^{-1}\pi + SP'(PSP')^{-1}(q - P S\Sigma^{-1}\pi) \quad (C6)$$

This vector of expected returns satisfies condition (C1) for any choice of the covariance matrix $S$. The corresponding GRE is:

$$D(q, m, p) = \frac{1}{2}(q - P S\Sigma^{-1}\pi)'(PSP')^{-1}(q - P S\Sigma^{-1}\pi) + \frac{1}{2}\text{Tr}(\Sigma S^{-1}) - n - \ln(|\Sigma S^{-1}|)) \quad (C7)$$

Two remarks need be made:

(i) In the absence of views on expected returns (i.e., no condition (C1), or $P = 0$) the forecast of expected returns should be, according to (C6),

$$p = S\Sigma^{-1}\pi \quad (C8)$$

In other words, a view on covariances should lead to an active portfolio with non-linear linear pay-offs only. The expected return vector in (C8) is the vector that is compatible with the existing linear market portfolio when the covariance forecast is $S$ instead of $\Sigma$. The corresponding GRE is:

$$D(q, m, p) = \frac{1}{2}\text{Tr}(\Sigma S^{-1}) - n - \ln(|\Sigma S^{-1}|)) \quad (C9)$$

(ii) With $S = \Sigma$ and views on some expected returns, the vector of expected returns is

$$p = \pi + \Sigma P'[P\Sigma P']^{-1}[q - PP\pi] \quad (C10)$$
as with the BLC model (re (A6)). The matching optimal active allocation is obtained by substituting (C10) into (B9):

\[ f_{q,m,p}(\mathbf{r}) = (\mathbf{q} - \mathbf{P} \pi \Sigma^{-1} \mathbf{P}')^{-1} \mathbf{P} \]  

(C11)

It is a linear combination of returns with asset weights:

\[ \mathbf{\omega} = \mathbf{P}'(\mathbf{P} \Sigma \mathbf{P}')^{-1} (\mathbf{q} - \mathbf{P} \pi) \]  

(C12)

The corresponding gain in certainty equivalent is, according to (B10):

\[ (1/\gamma) D(q, m, p) = \frac{1}{2}(\mathbf{q} - \mathbf{P} \pi)'(\mathbf{P} \Sigma \mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P} \pi) \]  

(C13)

Recovering the BLC results in this case is not surprising since the GRE distance under the condition \( S = \Sigma \) reduces to the quadratic distance \( \frac{1}{2}(\mathbf{p} - \pi)'\Sigma^{-1}(\mathbf{p} - \pi) \) that is minimized under the constraint (C1) in BLC; the LD method justifies the choice of objective function in BLC.

**C2 – Views on Expected Returns but no Views on some Covariances**

When the covariance matrix \( \mathbf{S} \) is not completely determined by personal views, we can minimize the Lagrangian (C4) over the choice of parameters in both \( \mathbf{p} \) and \( \mathbf{S} \). The minimisation over \( \mathbf{p} \) leads to (C6) and we are left with minimising the GRE (C7) over \( \mathbf{S} \) subject to condition (C2). It is easy to see that even with total absence of views on correlations, the choice \( \mathbf{S} = \Sigma \) minimizes the term between square brackets in (C7) but generally does not minimize the leading quadratic expression and therefore is generally not optimal. Thus, with the covariance matrix not fully determined, a view on some expected returns different from the market forecast should induce not only an adjustment of other expected returns but also an adjustment of the covariance matrix away from the market covariance matrix.

An analytical solution to this general GRE minimization problem requires solving a set of simultaneous non-linear equations that may rapidly become intractable. A general solver is best suited to this task.

We can illustrate analytically the fact that the choice \( \mathbf{S} = \Sigma \) is generally not optimal when conditioning on some expected returns by revisiting the single risky asset example at the end of Appendix B. There, an investor with a risk aversion coefficient \( \gamma = 4 \) can invest in a single risky asset with excess return of 8% and volatility of 20% according to the market forecast. We found that if he agrees with the market forecast, he should allocate half of his wealth to the risky asset and the other half to the risk free asset, but if his personal view was that the excess return should be only 6% and he still trusts a volatility of 20%, he should put in place an active portfolio of \(-1/8\), that is, allocate only 3/8th to the risky asset. That would increase his certainty equivalent by 1.125%. But suppose now that the investor is willing to
let the volatility forecast be adjusted away from 20%, what would be his new forecast, the corresponding active pay-off, and the increase in certainty equivalent?

In this univariate example the GRE (C7) reduces to:
\[
D(q, m, p) = \frac{1}{2}(p - (s/\sigma)^2/s^2 + \frac{1}{2}[(\sigma/s)^2 - 1 - 2\ln(\sigma/s)]
\]
Setting the derivative with respect to s to zero yields the solution
\[
s^2 = \frac{1}{2} (\sigma^4/\pi^2)[ -1 + (1 + 4(\sigma^2 + q^2)\pi^2/\sigma^4)^{0.5}] 
\]
Note that s = σ is a solution only when q = π. With the personal forecast p = 6%, volatility should be adjusted down from 20% to s = 19.46%. The corresponding active portfolio would be
\[
f_{q,m,p}(r) = -0.104r - 0.176 r^2 + 0.007
\]
and the gain in certainty equivalent would be only 0.10%. Thus, allowing for volatility adjustment, the active linear portfolio is less short in the risky asset (–0.104 instead of –0.125) but contains a short volatility position.

This finding is not immediately intuitive. However we note that when r = p, the sensitivity of the active portfolio to interest rate changes is –0.125, same as when s = σ. This is a general result that can be seen immediately from (B9). Assuming S = Σ leads to a linear active portfolio with weights (i.e., first order sensitivities)
\[
\omega = (\gamma \Sigma^{-1})(p - \pi)
\]
When S may be adjusted, the same first order sensitivities are obtained when the returns are at their expected values, but not elsewhere because of the non-linear component of the active pay-off.

**APPENDIX D – BLENDED FORECASTS**

Consider a risk neutral forecast q(r), a market forecast m(r) and a personal forecast p(r); we define as best compromise forecasts b(r), the solutions to:

\[
\text{Argmin}(b): D(q, p, b) \\
\text{Subject to: } D(q, m, b) = \text{constant} \tag{D1}
\]

where D(., .) is the GRE distance between the relevant probability densities.

(D1) can be solved analytically when q(r), m(r) and p(r) are multivariate normal distributions, then b(r) should also be a multivariate normal distribution. Using parameters N(0, Σ), N(π, Σ), N(p, S) and N(b, B) respectively for q(r), m(r), p(r) and b(r), we minimise the Lagrangian of the constrained minimization problem:

\[
\text{Argmin}(b, B): L = D(q, p, b) + \lambda[D(q, m, b) - \text{constant}] \tag{D2}
\]
Using (B10) for the GRE distances, the Lagrangian is such that:

\[
2L = (b - BS^{-1}p)'B^{-1}(b - BS^{-1}p) + Tr(SB^{-1}) - n - ln(|SB^{-1}|)
+ \lambda[(b - BS^{-1}p)'B^{-1}(b - BS^{-1}p) + Tr(SB^{-1}) - n - ln(|SB^{-1}|) - constant]
\]  (D3)

Setting the differential of L with respect to \(b\) equal to zero gives:

\[
B^{-1}(b - BS^{-1}p) + \lambda B^{-1}(b - B\Sigma^{-1}\pi) = 0
\]

for any \(B\), so that:

\[
b = B(S^{-1}p + \lambda\Sigma^{-1}\pi)/(1 + \lambda)
\]  (D4)

but of course, \(B\) can always be expressed as

\[
B = B(S^{-1}S + \lambda\Sigma^{-1}\Sigma)/(1 + \lambda)
\]  (D5)

Therefore the parameters of \(b\) and \(B\) of the best compromise forecast can be obtained by interpolation between the corresponding parameters of the market and personal forecasts using the same weights.
REFERENCES
EXHIBIT 1: LD WITH MARKET COVARIANCES ON TWO-RISKY ASSET PORTFOLIO

<table>
<thead>
<tr>
<th>Market (M) and Personal Views (P) on Expected Returns (%)</th>
<th>Posterior Expected Returns (%)</th>
<th>Optimal Allocations (%) and Gains in CE</th>
</tr>
</thead>
<tbody>
<tr>
<td>M p_A = 4.5, p_B = 10.5</td>
<td>4.50 10.5</td>
<td>50.0 50.0 0.0 0.000</td>
</tr>
<tr>
<td>P1 p_A = 5.5, p_B = 10.5</td>
<td>5.50 10.5</td>
<td>91.0 37.2 -28.2 0.205</td>
</tr>
<tr>
<td>P2 p_A = 5.5</td>
<td>5.50 11.75</td>
<td>75.0 50.0 -25.0 0.125</td>
</tr>
<tr>
<td>P3 p_B - p_A = 5.0</td>
<td>4.40 9.4</td>
<td>60.0 40.0 0.0 0.050</td>
</tr>
<tr>
<td>P4 11p_B - 21p_A = 0</td>
<td>4.94 9.43</td>
<td>81.7 33.4 -15.1 0.160</td>
</tr>
<tr>
<td>P5 50% P1 and 50% M</td>
<td>4.95 10.85</td>
<td>63.8 47.8 -11.7 0.025</td>
</tr>
<tr>
<td>P6 Interpol. (M, P1)</td>
<td>5.0 10.5</td>
<td>70.5 43.6 -14.1 0.050</td>
</tr>
</tbody>
</table>

**Posterior Expected Returns (%) with Market Covariances**

**Optimal Allocations (%) and CE Gains (bp) with Market Covariances**
EXHIBIT 2: LD WITH ADJUSTED COVARIANCES ON TWO-RISKY ASSET PORTFOLIO

<table>
<thead>
<tr>
<th>Market (M) and Personal Views (P) on Expected Returns (%)</th>
<th>Posterior Expected Returns (%)</th>
<th>Optimal Allocations (%) and Gains in CE</th>
</tr>
</thead>
<tbody>
<tr>
<td>M p_A = 4.5, p_B = 10.5</td>
<td>4.50 10.5</td>
<td>50.0 50.0 0.0 0.000</td>
</tr>
<tr>
<td>P1 p_A = 5.5, p_B = 10.5</td>
<td>5.50 10.5</td>
<td>79.3 39.7 -19.0 0.150</td>
</tr>
<tr>
<td>P2 p_A = 5.5</td>
<td>5.50 11.92</td>
<td>64.9 50.0 -14.9 0.077</td>
</tr>
<tr>
<td>P3 p_B − p_A = 5.0</td>
<td>4.35 9.35</td>
<td>57.3 42.7 0.0 0.036</td>
</tr>
<tr>
<td>P4 11p_B − 21p_A = 0</td>
<td>4.95 9.45</td>
<td>74.7 37.1 -11.7 0.123</td>
</tr>
<tr>
<td>P10 P1 with 62.5% correl</td>
<td>5.50 10.5</td>
<td>82.8 41.2 -24.0 0.17</td>
</tr>
</tbody>
</table>

EXHIBIT 3: ADJUSTED COVARIANCES AND COEFFICIENTS OF QUADRATIC PAY-OFFS

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Vol A (%)</th>
<th>Vol B (%)</th>
<th>Correl(A,B) (%)</th>
<th>r_A^2 (A straddles)</th>
<th>r_A r_B</th>
<th>r_B^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>10.00</td>
<td>20.00</td>
<td>62.50</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>P1</td>
<td>10.36</td>
<td>19.95</td>
<td>63.88</td>
<td>-0.84</td>
<td>-0.23</td>
<td>0.18</td>
</tr>
<tr>
<td>P2</td>
<td>10.32</td>
<td>20.46</td>
<td>64.63</td>
<td>-0.37</td>
<td>-0.31</td>
<td>0.00</td>
</tr>
<tr>
<td>P3</td>
<td>9.97</td>
<td>19.62</td>
<td>61.84</td>
<td>-0.16</td>
<td>0.02</td>
<td>0.13</td>
</tr>
<tr>
<td>P4</td>
<td>10.18</td>
<td>19.62</td>
<td>62.63</td>
<td>-0.65</td>
<td>-0.08</td>
<td>0.21</td>
</tr>
<tr>
<td>P10</td>
<td>10.19</td>
<td>19.63</td>
<td>62.50</td>
<td>-0.75</td>
<td>0.00</td>
<td>0.19</td>
</tr>
</tbody>
</table>
ENDNOTES

1 For short, throughout this paper we use the term 'return' to mean 'excess return' above the appropriate risk free rate over the investment horizon. The 'risk-free' asset depends naturally on the choice of objective (e.g., final wealth or buying power) and the investment horizon.

2 A quantitative expression of risk attitude is not even necessary if one is concerned about selecting a portfolio of risky assets only, leaving the choice of leveraging to investors.

3 Analysts often go to extraordinary lengths to avoid making quantitative probabilistic forecasts, whether for fear of being challenged or for fear of being misunderstood. Most prefer to use qualitative forecasts or to make direct investment recommendations (such as “buy” or “hold”) although they may not know fully their clients’ positions, objectives and constraints.

4 Even when the objective of portfolio management depends on a single characteristic of the return distribution, for example a value-at-risk constraint with severe penalties, there is no point in delving into the uncertainty in the evaluation of that characteristic. This uncertainty should bear no relevance to allocation decisions. Take a more vivid example: if I had to cross a flimsy bridge for the pleasure of exploring the other side of a precipice, I would balance those pleasures against the inconvenience of falling to my death should the bridge break. I would therefore assess with great care the probability of the bridge collapsing under my weight. But whatever uncertainty there may be about that probability, it should not affect in my decision to cross the bridge or not.

5 It is more likely that forecasts of expected returns would be based on information pertaining to the future rather than historical price series, for example in the case of a share price, information about a change of management, new contracts, new technologies, pending lawsuits, etc.

6 Even with constant trends and volatilities – and there is little empirical evidence for that in asset returns – statistical trend estimates are highly uncertain. The statistical error (standard deviation) for an average return based on a series of n iid observations with volatility $\sigma$ is equal to $\sigma/\sqrt{n}$. For example, to estimate with a 1% statistical error the yearly trend of a price series with 20% volatility would require 400 years of observations.

7 We use the following notations: bold lower case letters for column vectors and bold upper case letters for matrices. An apostrophe denotes a transpose and a $(-1)$ exponent marks an inverse. Otherwise we keep as closely as possible to the notations in Black and Litterman [1992].

8 This relationship results from the unconstrained maximization of the certainty equivalent $\pi\omega - (\gamma/2)\omega'\Sigma\omega$

9 With continuous time price models and continuous observations, deterministic volatilities and correlations, would be perfectly known instantaneously. For estimation errors when these assumptions are relaxed, see Jobson and Korkie[1980]). Some estimation methods use structural models to reduce the number of correlation parameters (e.g., linear factor models) and shrinkage techniques to obtain robust estimates (Jorion [1986]).

10 Option prices yield implied volatilities and correlations over their tenor, though correlation sensitive instruments – spread options, basket options, exchange options, etc. – are often illiquid and give therefore only approximate estimates of correlations. Options implied volatilities and correlations are especially relevant when these instruments are included in the universe of assets available for investment.

11 The average risk attitude of a group of investors is difficult to estimate. One would need to know their total wealth (including the evaluation of human capital, illiquid assets, etc.) and how it is allocated. Due to lack of information and heterogeneity among investors, one finds illustrative figures for risk aversion coefficients in the literature ranging from less than 2 to more than 6.

12 Only if expected returns were estimated statistically from historical price series, which they are not in this instance, would the asymptotic distribution of estimates be multivariate normal with covariance matrix $\Sigma/\sqrt{n}$.

13 Strictly speaking, this is only true with some types of probability distributions.

14 The risk neutral forecast corresponds to market implied covariances and zero expected returns for all assets and traded derivatives. In a complete market, it determines the market price of any derivative contract.

15 An exponential utility function can generally be used as a first approximation of other utility functions over a limited range of risks.

16 The relative entropy distance between two distributions with pdf’s $m$ and $p$, where the support of $m$ is included in the support of $p$, is defined as $D(m, p) = I_m[\ln(m/p)]$; it is used in many fields, especially in information theory where it is also known as the Kullback-Leibler divergence. The relative entropy distance is always non-negative and equal to zero if and only if $m = p$. But it is not a metric because it is not symmetric and does not satisfy the triangular inequality.

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