ROM Simulation: Applications to Stress Testing and VaR

Carol Alexander\textsuperscript{a} and Daniel Ledermann\textsuperscript{b}

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Abstract

Most banks employ historical simulation for Value-at-Risk (VaR) calculations, where VaR is computed from a lower quantile of a forecast distribution for the portfolio’s profit and loss (P&L) that is constructed from a single, multivariate historical sample on the portfolio’s risk factors. The implicit assumption is that history will repeat itself for certain over the forecast horizon. Until now, the only alternative is to assume the historical sample is generated by a multivariate, parametric risk factor distribution and (except in special cases where an analytic solution is available) to simulate P&L via Monte Carlo (MC). This paper introduces a methodology that encompasses historical and MC VaR as special cases, which is much faster than MC simulation and which avoids the single-sample bias of historical simulation. Random orthogonal matrix (ROM) simulation is a fast matrix-based simulation method that applies directly to an historical sample, or to a parametric distribution. Each simulation matches the first four multivariate sample moments to those of the observed sample, or of the target distribution. Stressed VaR is typically computed from an historical sample using the Duffie-Pan methodology, whereby the sample is transformed to have a stressed covariance matrix. ROM simulation extends this methodology to generate very large samples, which furthermore have stressed values for the first four multivariate moments values.

\textbf{JEL Codes:} C14, C15, C53, C63, G17, G21, G28

\textbf{Keywords:} Random orthogonal matrix, Value-at-Risk, Stressed VaR, Basel II, Market risk capital.

\textsuperscript{a} Chair of Risk Management, ICMA Centre, Henley Business School at Reading, Reading, RG6 6BA, UK. Email: c.alexander@icmacentre.ac.uk

\textsuperscript{b} Senior Analyst, Sungard, London. Email: daniel.ledermann@sungard.com
1. Introduction

Recent recommendations from the Basel Committee on Banking Supervision require a regulatory capital add-on to cover the extreme losses that are simulated by stress testing positions. This is to be done by adding to the standard capital charge, which is based on 1% 10-day Value-at-Risk (VaR), an additional charge based on a ‘stressed VaR’, i.e. the VaR that is computed for current positions assuming they are held during a period of extreme turmoil, such as the credit and banking crisis of 2007-2008. Most major banks employ historical simulation for VaR calculations and would therefore compute stressed VaR using the Duffie and Pan [1997] approach, whereby a stressed risk-factor covariance matrix is computed from the extreme sample and then imposed upon the larger historical sample. Since only one large historical sample is available there can be significant sample bias.

Random orthogonal matrix (ROM) simulation is a novel approach to simulation introduced by Ledermann et al. [2011]. It has applications to any problem where historical or Monte Carlo (MC) simulation is commonly applied as the method of resolution. ROM simulation is the same as historical or MC simulation (depending on whether the original sample is historical or MC generated) when the random orthogonal matrix is replaced by the identity matrix. But using random orthogonal permutation and/or rotational matrices in ROM simulation effectively re-samples from the original data to produce simulations that may exhibit a variety of characteristics, as described in Ledermann and Alexander [2012].

Being based on matrix multiplication rather than parametric distributions, ROM simulation is much faster than MC methods, and it has less simulation error because the $L$ matrix which is fundamental to ROM simulation is calibrated in such a way that the first four multivariate sample moments are matched to target multivariate moments. ROM simulation also alleviates the problems of data limitation and sample bias in historical simulation. It is different from the statistical bootstrap, which randomly re-samples from the same basic data. ROM simulation introduces additional uncertainty by repeatedly applying to the original sample different random orthogonal matrices, of a type with known ROM simulation characteristics, thus generating new random samples that have multivariate moments consistent with those of the original sample.

This paper explains how ROM simulation can be used to compute VaR and stressed VaR. An empirical study demonstrates that ROM VaR can produce more accurate unconditional VaR model estimates than standard unconditional VaR models. Furthermore, it can quantify the sample bias in the Duffie and Pan [1997] (D-P) approach, extend their stressed-covariance methodology to stressed levels of multivariate skewness and kurtosis, and produce any number of simulated samples that are consistent with the stressed multivariate moments. Its advantages are: that one is able to stress test a portfolio for an increase
in extreme returns in addition to breakdown in correlations; one avoids the single-sample bias of the D-P approach; and because matrix multiplication is so fast the algorithm is much faster than parametric simulation methods.

We proceed as follows: Section 2 provides an overview of the literature on VaR and stress testing that is most relevant to our work, and describes the recent changes to banking regulations on market risk; Section 3 provides a general introduction to ROM simulation; Section 4 explains the application of ROM simulation to VaR and stress VaR; Section 5 presents our empirical study and Section 6 summarizes and concludes.

2. VaR and Stressed VaR

Multivariate distribution forecasting is fundamental to portfolio risk assessment. Parametric forecasting methods for portfolio returns or profit and loss (P&L) target a multivariate distribution for the portfolio’s risk factors with parameters that are based on the analyst’s beliefs. Non-parametric distribution forecasting methods take an historical sample on the risk factors and use the empirical distribution of this sample to represent the forward-looking portfolio returns or P&L distribution. The Value-at-Risk (VaR) of a portfolio is the loss that would be equalled or exceeded, with a given probability $\alpha$, over a certain period of time (the ‘risk horizon’) if the portfolio is left unmanaged during this interval. Measured in nominal terms, the $\alpha\%$ $h$-day VaR is minus the $\alpha$-quantile of the portfolio’s $h$-day P&L distribution. The portfolio mapping is applied to multivariate risk factor scenarios to simulate a univariate $h$-day P&L distribution for the portfolio and the $\alpha\%$ VaR is then estimated from its $\alpha$-quantile. In a survey of large commercial banks by Perignon and Smith [2010], 73% of the responding banks applied historical simulation to risk factor returns, 22% used MC simulation and the other 5% mostly used some type of hybrid simulation method for computing VaR for market risk capital requirements.

The MC and historical simulation approaches have different advantages and limitations. Historical VaR builds portfolio P&L distributions using a large sample of observed risk factors. It does not make parametric assumptions regarding the risk factors – it allows

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1Portfolio risk factors include returns on equity indices for stock portfolios, exchange rate returns for international portfolios, changes in interest rates for fixed income portfolios and returns on commodity futures. The analyst’s beliefs could be objective (as when parameter estimates are derived from an historical sample on the risk factors) or subjective (as in scenario analysis) or a combination of the two (as in Black and Litterman [1992] for instance).

2Only when the portfolio is a linear function of multivariate normally distributed risk factors does an analytic solution for the $\alpha$-quantile exist. However, since Mandelbrot [1963] and Fama [1984] it has been widely accepted that the assumption of multivariate normal distributions for financial risk factors is not empirically valid, except perhaps when risk factors are sampled at very low frequency; and many portfolios are non-linear functions of their risk factors. Thus few, if any, large banks would employ an analytic VaR method for risk capital calculation.
the data to speak for itself, and this is thought to be advantageous for capturing complex risk factor co-dependencies that are difficult to capture parametrically. Another advantage is that it is relatively easy to implement. However, historical simulations are backward looking and implicitly assume that the distribution derived from the observed sample will be realised for sure over the risk horizon. To obtain sufficient precision in the extreme quantile estimates a large sample is required. But this poses the operational challenge of obtaining a large historical sample on all the risk factors and has the further disadvantage that the sample is likely to include periods where market conditions were quite different from those prevailing at the time the VaR is estimated.\(^3\) Trading-off this quantile inaccuracy with the risk-insensitivity associated with long historical samples, many academics and practitioners use sample sizes of around two–three years, or about 500–750 daily observations, for historical VaR estimation. The Basel committee on banking supervision recommend a minimum in-sample period of one year.

Hull and White [1998] tackle the risk-insensitivity issue by rescaling historical observations using generalised conditional heteroscedastic (GARCH) volatilities. For a time series of portfolio returns \(r_t\), with \(0 < t \leq T\), GARCH variances \(\hat{\sigma}_t^2\) are estimated. Then, using the most recent volatility forecast \(\hat{\sigma}_T\), the returns are scaled as \(r_t^* = r_t \hat{\sigma}_T \hat{\sigma}_t^{-1}\). If the GARCH model is well-specified then \(r_t^*\) should have the constant variance \(\hat{\sigma}_T\) for \(0 < t \leq T\). VaR is then estimated from these scaled returns in the usual way. Barone-Adesi et al. [1998] extend this approach to derive VaR from a portfolio distribution that is simulated from the estimated GARCH model itself, with innovations re-sampled from the historical returns rather than simulated from some parametric distribution for the GARCH model innovations. Boudoukh et al. [1998] propose that risk factor returns are scaled by applying exponentially declining weights to past returns, to obtain an age-weighted empirical P&L distribution for the portfolio, in which the most recent data are given the largest weights.

However, Pritsker [2006] claims that historical VaR estimates fail to respond fast enough to new market conditions even when historical samples are weighted or filtered using the methods just described. A different conclusion is reached by Alexander and Sheedy [2008], who demonstrate that the Barone-Adesi et al. [1998] GARCH filtered historical simulation can provide accurate VaR and conditional VaR estimates, even at very extreme quantiles.

Data limitation is not an issue with MC VaR, but sampling error can be a serious problem that is usually resolved by using a very large number of simulations on the risk factors. Most banks regard 10,000 Monte Carlo simulations as an expedient minimum, but recognise that an acceptable degree of accuracy can only be achieved with many more

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\(^3\)For instance, just after the turn of the century a large historical sample would reflect predominately stable and upward trending equity markets, whereas the first quarter of the year 2000 marked the beginning of a highly uncertain era, especially for technology stocks.
simulations, plus the application of advanced sampling techniques. See Glasserman et al. [2000, 2001] and Glasserman [2004] for further details. Thus, when positions include large, complex portfolios which themselves require simulation for mark-to-model valuation, it can be too time-consuming to be useful except for overnight VaR computations. Even with very powerful computers the intra-day VaR computations required to assess trader’s risk limits can only be applied by substituting simplified risk factor mappings for complex portfolios.

Another problem with MC VaR is that different distributional assumptions for the risk factors typically give rise to quite different MC VaR estimates. The empirical characteristics of financial asset returns are not always well represented by multivariate normal, Student-t or even normal mixture distributions, yet it is usually not clear what distributional assumption is most appropriate, even for standard risk factors such as equity index returns. The existence of volatility clustering is widely accepted, so that conditional returns distributions (e.g. based on a multivariate GARCH risk-factor returns process) would be theoretical justified. According to Alexander and Sheedy [2008] such conditional VaR models produce estimates that are at least as accurate as the filtered historical simulation approach.

Despite the demonstrated success of conditional VaR models in predicting VaR at the portfolio level, there are major impediments to the implementation of such models in a large corporation. For internal, economic capital allocation purposes VaR models are commonly built using a ‘bottom-up’ approach. That is, VaR is first assessed at an elemental level, e.g. for each individual trader’s positions, then is it progressively aggregated into desk-level VaR, and VaR for larger and larger portfolios until a final VaR figure for a portfolio that encompasses all the positions in the firm is derived. This way, risk budgets at all levels of the firm’s activities can be based on a unified risk-assessment framework. But it could take many days to compute the full (often MC simulated) valuation models for each product under each of the scenarios generated in the VaR model. For regulatory purposes VaR must be computed at least daily, and for internal risk-based management intra-day VaR computations are frequently required. Therefore, to increase the speed of VaR calculations banks impose a major reduction in complexity of both valuation models and the VaR model itself. Conditional VaR models are far too complex, econometrically and computationally, to be integrated into an enterprise-wide risk assessment framework.

The other reason why banks avoid the use of conditional VaR models is that the market risk capital allocations that are derived from them would change too much over time. Risk budgeting, from the setting of trader’s limits to internal economic capital allocation for broad classes of activities, is typically based on decisions that are made infrequently, e.g. at monthly or quarterly meeting among chief risk officers and other senior managers. If risk assessment were based on a conditional VaR model then trader’s would find their limits being
exceeded on a very regular basis, and hence the economic capital limits at desk level and more aggregate levels would also be exceeded too often. Moreover, there are systemically dangerous, pro-cyclical effects when banks base their minimum required reserves of risk capital on a conditional VaR framework. See Flannery et al. [2012] and Longbrake and Rossi [2011] for further information. This is another reason why the majority of banks still base VaR estimates on unconditional (parametric or non-parametric) distributions for risk factor returns.

How would the adoption of ROM simulation resolve some or all of these problems associated with existing VaR resolution methods? ROM simulation provides unconditional VaR estimates that can be based on arbitrarily large samples that are designed to be consistent with historically observed or target multivariate moments up to fourth order. So it does not suffer from the single-sample bias of historical simulation – instead it generates any number of forward-looking scenarios that are consistent with the salient features of a single historical sample, i.e. the features that are captured by its moments. Likewise, when based on general target moments, it does not suffer from the main constraint of standard MC simulation, i.e. that all scenarios are based on a parametric distribution that is assumed to be known. Furthermore, ROM simulation could be implemented as a relatively simple add-on to the existing VaR system, by augmenting the simulations generated by either historical or MC simulation VaR models, with the purpose of counteracting their in-built limitations.

Under the original Basel II Accord the minimum required capital (MRC) for a portfolio in a commercial bank equals the maximum of (a) its most recent 1% 10-day VaR estimate and (b) the average of the previous sixty 1% 10-day VaR estimates multiplied by a scaling factor \( m_c \) of between 3 and 4 whose value depends on the VaR model’s performance in regulator’s backtests. Even though 10-day VaR estimates could be obtained directly using MC (or ROM) simulation, historically simulated 10-day forecasts would be seriously limited by the available data. Hence it is industry standard to scale daily VaR estimates up by a factor of \( \sqrt{10} \), invoking the industry-standard ‘square-root-of-time’ rule. Then, given a series \( \text{VaR}_{t-i} \) of 1% daily VaR estimated on day \( t - i \), the daily MRC at time \( t \) was typically computed as:

\[
MRC_t = \sqrt{10} \times \max \{ \text{VaR}_{t-1}, m_c \times \frac{1}{60} \sum_{i=1}^{60} \text{VaR}_{t-i} \},
\]

(1)

However, the experience of the 2008 banking crisis showed that VaR estimates were not conservative enough. In response, the Basel Committee recommended that revisions to the Basel II market risk framework in July 2009 include the computation of stressed VaR (sVaR) as a measure ‘intended to replicate a VaR calculation that would be generated on the bank’s current portfolio if the relevant market factors were experiencing a period of stress’. Thus,
sVaR is minus the 1% quantile of hypothetically stressed 10-day portfolio P&L. Although no specific model is prescribed, the inputs to an sVaR calculation must be calibrated to at least 12 months of stressed historical data. Hence, the Basel committee now recommends that MRC be based on both VaR and sVaR estimates, via the following calculation:

$$MRC_t = \max \left\{ \text{VaR}_{t-1} - 1, mc \times \frac{1}{60} \sum_{i=1}^{60} \text{VaR}_{t-i} \right\} + \max \left\{ \text{sVaR}_{t-1} - 1, ms \times \frac{1}{60} \sum_{i=1}^{60} \text{sVaR}_{t-i} \right\}.$$ (2)

Just like $mc$, the multiplicative constant $ms$ is also subjected to a minimum value of 3.

Practitioners are drawn to the historical approach to stress testing because complex dependency structures, which are typically further complicated during a financial crisis, can be used to analyse current exposures. However, data limitations remain a serious issue. One of the great advantages of ROM simulation is that it can be used with an extreme historical sample of any size to simulate as much stressed risk factor data as we like. Furthermore, the multivariate moments of the observed extreme sample will be preserved in the much larger ROM simulated sample. Skewness and kurtosis are particularly important to target in stressed VaR calculations, since stressing these values will increase the probability of extreme risk factor movements.

When applied to compute stressed VaR, ROM simulation can be regarded as a generalization of the Duffie and Pan [1997] methodology for modifying a long time series of historical risk factor data $X_{mn}$, where $m$ is the number of observations and $n$ is the number of risk factors, to reflect extreme market conditions. When $m$ is large the sample $X_{mn}$ will typically contain data from tranquil periods as well as crisis periods and will not epitomise a stressed sample. To overcome this, D-P suggest a simple method for transforming $X_{mn}$ into a stressed sample $\tilde{X}_{mn}$. They first construct a stressed covariance matrix $\tilde{S}_n$ which may correspond to a shorter, but stressed, period of historical returns or be specified hypothetically. Either way, the corresponding stressed sample is defined as:

$$\tilde{X}_{mn} = 1_m \mu_n' + (X_{mn} - 1_m \mu_n') A_n^{-1} \tilde{A}_n,$$ (3)

with $\tilde{A}_n' \tilde{A}_n = \tilde{S}_n$ and $A_n' A_n = S_n$, where $S_n$ is the covariance matrix and $\mu_n$ is the mean vector of $X_{mn}$. It is straightforward to check that the covariance matrix of the adjusted sample is indeed the stressed covariance matrix $\tilde{S}_n$, and the mean is unchanged. Furthermore, $\tilde{X}_{mn}$ will display co-dependency characteristics of actual risk factor returns, since it is a linear transformation of the historical sample $X_{mn}$. The main advantage of this construction is that the stressed sample contains $m$ observations, so quantiles associated with stressed VaR calculations may be estimated with greater precision.
However, the D-P methodology relies on the assumption that covariance matrices capture all the features necessary for stress testing. Higher order moments such as skewness and kurtosis are not taken into account: the Mardia [1970] skewness and kurtosis of the stressed sample based on (3) are identical to unstressed higher sample moments, since (3) is an affine transformation. The same critique applies to stress tests based on MC simulation since, for consistency with front-office trading activities, most models used in practice assume the risk-factor distribution is elliptical—see Alexander and Sarabia [2012]. Yet these higher moments are likely to change significantly during a crisis period, and stress testing portfolios for an increase in extreme returns (especially those on the downside) should be one of the main priorities for computing stressed VaR. ROM simulation addresses this problem by constructing a sample which is consistent with the higher moments of stressed market data, or target values of these moments that are otherwise chosen to reflect a period of stress. Because ROM simulation is an encompassing method, the covariance matrix may also be stressed at the same time, if desired. Furthermore, since ROM simulation can generate as many scenarios as we like in a very rapid timeframe, stress tests can be based on samples that are large enough to estimate extreme quantiles very reliably.

3. Overview of Random Orthogonal Matrix Simulation

ROM simulation was introduced by Ledermann et al. [2011] as a method to generate random samples that always have the same, target sample mean vector and sample covariance matrix and the same multivariate skewness and kurtosis. It is most easily understood by considering how to adjust a random sample \(X_{mn}\) of size \(m\) from an \(n\)-dimensional multivariate distribution so that the sample mean vector and sample covariance matrix exactly matches some target mean vector \(\mu_n\) and covariance matrix \(S_n\). Since \(S_n\) must be positive semi-definite, we can always find a decomposition of the form \(S_n = A_n'A_n\).\(^4\) Now consider the following transformation of \(X_{mn}\):

\[
L_{mn} = m^{-1/2}(X_{mn} - 1_m\mu_n')A_n^{-1}.
\]

Clearly, the mean of \(X_{mn}\) is \(\mu_n\) and its covariance matrix is \(S_n\) if, and only if:

\[
L_m'n'L_{mn} = I_n \text{ with } 1_m'L_{mn} = 0_n'.
\]

\(^4\)For instance, \(A_n\) could be a Cholesky decomposition of \(S_n\) or we could set \(S_n = Q_n\Lambda_nQ_n'\), where \(\Lambda_n\) is the diagonal matrix of eigenvalues, and \(Q_n\) is the orthogonal matrix of eigenvectors of \(S_n\), so that \(A_n = \Lambda_n^{1/2}Q_n'\).
Any \( m \times n \) orthogonal matrix \( L_{mn} \) satisfying (5) is called an \( L \) matrix. Ledermann et al. [2011] introduced three distinct classes of such matrices (deterministic, parametric, and data specific) which are briefly described below. Now, given any \( L \) matrix \( L_{mn} \) satisfying (5) we can invert the transformation (4) to obtain an exact mean and covariance sample, in the form:

\[
X_{mn} = 1_m \mu'_n + m^{1/2}L_{mn}A_n.
\]  

(6)

One of the essential properties of \( L \) matrices, which follows immediately from the definition (5), is that if \( L_{mn} \) is an \( L \) matrix then so is \( Q_mL_{mn}R_n \) where \( Q_m \) is an \( m \times m \) permutation matrix and \( R_n \) is a general \( n \times n \) orthogonal matrix. The fundamental idea of ROM simulation is to use random elements in these matrices. That is, starting with an \( L \) matrix \( L_{mn} \) and a random orthogonal matrix \( R_n \), ROM simulation generates random samples \( X_{mn} \) via the equation:

\[
X_{mn} = 1_m \mu'_n + m^{1/2}Q_mL_{mn}R_nA_n,
\]  

(7)

where \( Q_m \) is a random permutation matrix, and \( A'_nA_n = S_n \).

While ROM simulation preserves the multivariate (Mardia) skewness and kurtosis, since these measures are invariant under orthogonal transformations, the skewness and kurtosis of the marginal distributions are changed under ROM simulation. Ledermann and Alexander [2012] explain how the marginal’s characteristics are altered by ROM simulation when using random (a) upper Hessenberg, (b) Cayley and (c) exponential rotational matrices for \( R_n \). They show that exponential matrices reduce the central mass of the marginals, relative to that of the basic \( L \) matrix, whereas upper Hessenberg and Cayley matrices tend to increase the central mass. Exponential and Cayley matrices induce positive skew when upper Hessenberg induce negative skew, and conversely. Ledermann et al. [2011] explain how random permutation matrices \( Q_n \) alter the dynamic properties of the sample: essentially, arbitrary permutation matrices change the autocorrelation properties, cyclic permutation matrices change the timing of volatility clusters. Reflection matrices \( R_n \) control the sign of the marginal’s skewness and reflections can be added to switch a negative marginal skewness to a positive skewness of equal magnitude, and vice versa.

\( L \) matrices can be found by orthogonalising different linearly independent sets within

\(^5\)If the columns of a matrix \( Y_{mn} \) sum to zero, then the columns of the product \( Q_mY_{mn} \) will also sum to zero if \( Q_m \) is a permutation matrix, but otherwise this is not necessary true. Thus \( L \) matrices can be pre-multiplied by permutations, while general orthogonal matrices (rotations and reflections) can only be used for post-multiplication. Further properties of permutation matrices with regard to ROM simulation, and those of reflection matrices, are discussed in Ledermann et al. [2011]. The ROM simulation characteristics of rotational matrices are fully explored in Ledermann and Alexander [2012].

\(^6\)For instance, with Ledermann or Type I or Type II \( L \) matrices, Hessenberg ROM simulations have negative skew and exponential and Cayley ROM simulations have positive skew; the opposite is the case with Type III \( L \) matrices.
the hyper-plane $\mathcal{H} \subseteq \mathbb{R}^m$, defined by

$$\mathcal{H} = \{(h_1, \ldots, h_m)' \in \mathbb{R}^m \mid h_1 + \cdots + h_m = 0\}.$$  \hfill (8)

Typically, solutions are constructed in two steps: (1) Take a pair $(m, n) \in \mathbb{Z}_+^2$ with $m > n$ and pick $N(m)$ linearly independent vectors in $\mathcal{H}$, where $m > N(m) \geq n$. Use these vectors to form the columns of a matrix $V_{m,N(m)}$; (2) Apply the Gram-Schmidt (GS) procedure to $V_{m,N(m)}$. This produces a matrix $W_{m,N(m)}$, with orthonormal columns. Then select $n$ columns from $W_{m,N(m)}$ to form a matrix $L_{mn}$.

The properties of an $L$ matrix are inherited from the linearly independent vectors used in its construction. Deterministic $L$ matrices use any set of deterministic vectors satisfying (5), which in particular includes the Ledermann matrix $L_{mn} = (\ell_1, \ldots, \ell_n)$, where

$$\ell_j = [(m - n + j - 1)(m - n + j)]^{-1/2}(1, \ldots, 1, -(m - n + j - 1), 0, \ldots, 0)' \hfill (9)$$

for $1 \leq j \leq n$. Introducing a further, positive integer parameter $k$ to define deterministic $L$ matrices allows ROM simulation to target multivariate skewness and kurtosis, as defined by Mardia [1970]. Ledermann et al. [2011] introduce this parameter in three different ways, thus defining Type I, Type II and Type III $L$ matrices.

For brevity, this paper will consider only Type I deterministic $L$ matrices, which are constructed as follows: set $N(m) = m + 1 - 2k$ and, to ensure that $n \leq N(m)$, we require $2k \leq m + 1 - n$. Then set $V_{m,N(m)} = (v_1, \ldots, v_{N(m)})$, where

$$v_j = [0, \ldots, 0, 1, -1, \ldots, 1, -1, 0, \ldots, 0]' \quad \text{for} \quad 1 \leq j \leq N(m). \hfill (10)$$

We also consider two other types of $L$ matrix for which ROM simulation can be interpreted as an extension of MC or historical simulation:

- In parametric $L$ matrices the columns of the GS pre-image matrix $V_{mn} = (v_1, \ldots, v_n)$ are random vectors drawn from a zero mean elliptical multivariate distribution, whose marginal components are independent. First, a single MC simulation, adjusted to achieve exact covariance, is used to construct the $L$ matrix in (7) and then random permutation matrices $Q_n$ and/or random orthogonal matrices $R_n$ (rather than re-sampling from a parametric distribution) are applied in (7) to derive further samples with the same sample mean and covariance matrix. Ledermann and Alexander [2012] prove that when the original sample is MVN, the ROM simulations are also MVN distributed, and that ROM simulation can be hundreds (or even thousands) of times
faster than MC simulation.

- Data-specific $L$ matrices are formed from a linearly independent set in $\mathcal{H}$ taken directly from an observed sample. In this case ROM simulation becomes an extension of historical simulation. The repeated application of random permutation matrices $Q_n$ and/or random orthogonal matrices $R_n$ produces new random samples, via (7), which preserve the sample mean vector, covariance matrix and higher multivariate moments.

4. ROM VaR and Stressed VaR

Any vector representing $n$ risk factor returns determines a portfolio return via the mapping $\pi : \mathbb{R}^n \to \mathbb{R}$. Applying this map to a row-oriented matrix of ROM simulated scenarios $X_{mn} = (x_1', \ldots, x_m')'$ produces a sample of $m$ portfolio P&Ls, $\Delta \mathbf{P}_m = (\Delta P_1, \ldots, \Delta P_m)'$ where $\Delta P_i = \pi(x_i)$ for $1 \leq i \leq m$ and the change $\Delta$ is taken over the risk horizon for the VaR estimate. To calculate the VaR we first write the portfolio returns as order statistics $\Delta P_{(1)}, \ldots, \Delta P_{(m)}$, from smallest to largest. We then assign each of these values to quantile levels via the mapping $\Delta P_{(i)} \leftrightarrow (i - 0.5)/m, i = 1, \ldots, m$. To estimate an $\alpha$-quantile we find the value of $i$ satisfying $i - 0.5 \leq m\alpha \leq i + 0.5$, and then linearly interpolate between $\Delta P_{(i)}$ and $\Delta P_{(i+1)}$. The corresponding VaR estimate is then $-1 \times$ this $\alpha$-quantile

When computing stressed VaR we transform historical data $X_{mn}$ with covariance matrix $S_n$ into a stressed data sample $\tilde{X}_{mn}$ with covariance matrix $\tilde{S}_n$. The D-P approach is given by (3) above. In the context of historical ROM simulation, we first transform the historical sample $X_{mn}$ into a rectangular orthogonal matrix $L_{mn}^D$ using the Gram-Schmidt procedure. This data-specific $L$ matrix is then used to construct a stressed random sample via the transformation:

$$\tilde{X}_{mn} = 1_m\mu_n' + m^{1/2}L_{mn}^D R_n \tilde{A}_n,$$

where $\tilde{A}_n$ is a Cholesky decomposition of the stressed covariance matrix $\tilde{S}_n$. We now show that (11) is equivalent to (3) when $R_n$ is the identity matrix. It is sufficient to show that

$$L_{mn}^D = GS(X_{mn}) = m^{-1/2}X_{mn}A_n^{-1},$$

where $A_n$ is the Cholesky decomposition of $S_n$. Here we must assume that $S_n$ is positive definite, so that the Cholesky matrix $A_n$ is unique and has a strictly positive diagonal. It is possible, using standard matrix decomposition methods, to write $X_{mn} = L_{mn}^D U_n$, where $U_n$ is upper triangular with positive diagonal elements. Since $L_{mn}^D$ is rectangular orthogonal we deduce that

$$S_n = m^{-1}X_{mn}'X_{mn} = m^{-1}(L_{mn}^D U_n)'(L_{mn}^D U_n) = (m^{-1/2}U_n)'(m^{-1/2}U_n).$$
Now the uniqueness of the Cholesky decomposition dictates that $A_n = m^{-1/2}U_n$. This shows that constructions (3) and (11) are indeed equivalent when $R_n = I_n$.

The D-P method yields only one one stressed sample, so it is highly vulnerable to sample bias. ROM simulation allows one to generate many stressed samples, each consistent with the first four multivariate moments of the corresponding D-P sample and each yielding a different sVaR estimate. By examining the variability of ROM simulated sVaR estimates, calculated from different ROM simulations with identical means, covariances and multivariate skewness and kurtosis we shall investigate the sample bias associated with D-P sVaR estimates.

A limitation of the D-P method is that samples are only stressed through their covariances, yet it is widely accepted that higher moments also increase in magnitude during a financial crisis. Indeed, increasing the higher moments should be more important than stressing the covariance matrix when stress testing a portfolio. Using ROM simulation we can stress samples through their multivariate skewness and kurtosis as well as their covariance, by augmenting a transformation of the unstressed sample with additional ROM simulations based on an $L$ matrix that is chosen to target stressed levels of skewness and kurtosis. For instance, denote by $\kappa := \kappa_M(X_{mn})$ the multivariate Mardia [1970] kurtosis of the original (unstressed) sample $X_{mn}$ with covariance matrix $S_n$. The objective is to transform $X_{mn}$ into a stressed sample $\tilde{X}_{mn}$, with covariance matrix $\tilde{S}_n$ and multivariate kurtosis $\tilde{\kappa} := \kappa_M(\tilde{X}_{mn})$.

First we transform $X_{mn}$ into a data-specific $L$ matrix defined by (9), we form our stressed sample using a concatenation of the form:

$$\tilde{X}_{mn} = \begin{pmatrix} m^{1/2}L_{mn}^D \\ p^{1/2}L_{pn}R_n^1 \\ \vdots \\ p^{1/2}L_{pn}R_n^r \end{pmatrix} \tilde{A}_n,$$

where $R_n^1, \ldots, R_n^r$ are orthogonal matrices and $\tilde{A}_n$ is the cholesky matrix of our target stressed covariance matrix $\tilde{S}_n$.

Intuitively, this construction will augment an historical sample with ROM simulated data so that the kurtosis of the combined sample is increased to the target, stressed level of kurtosis. When $r = 0$ the construction reduces to the standard D-P stress test. In particular,

$$\kappa_M(\tilde{X}_{mn}) = \kappa_M(L_{mn}^D) = \kappa_M(X_{mn}) := \kappa.$$

However, when $r > 0$, this relationship will not hold. To find the multivariate kurtosis of
we apply Proposition 2.2 of Ledermann et al. [2011] to obtain the expression

$$\kappa_M(\tilde{X}_{mn}) = (m + rp)^{-1} \left[ nk + rp\kappa_M(L_{pm}) \right]. \quad (13)$$

We want to choose \( p \) so that \( \kappa_M(\tilde{X}_{mn}) \) is equal to the stressed kurtosis level \( \tilde{\kappa} \). By Proposition 2.1 of Ledermann et al. [2011] \( \kappa_M(L_{pm}) = n [(p - 2) + (p - n)^{-1}] \). Hence, using the approximation \( \kappa_M(L_{pm}) \approx n(p - 2) \) we can reduce (13) to a quadratic equation in \( p \). We are only interested in the positive solution of this quadratic, which is given by the formula:

$$p^* = \frac{(2n + \tilde{\kappa}) + \sqrt{(2n + \tilde{\kappa})^2 + 4mn(p - n)^{-1}(\tilde{\kappa} - \kappa)}}{2n}. \quad (14)$$

Note that \( p^* \) will certainly be positive if \( \tilde{\kappa} > \kappa \). If we now substitute the integer \( p = \text{int}(p^*) \) into (12) then the stressed sample \( \tilde{X}_{mn} \) will have multivariate kurtosis approximately equal to the stressed level \( \tilde{\kappa} \).

If instead of the Ledermann matrix above we use a \( p \times n \) Type I \( L \) matrix then we have an additional parameter \( k \) at our disposal, and we can choose this to target the multivariate skewness as well as the kurtosis. An example of this is given in the next section.

5. Empirical Study

5.1. Data

Our empirical study is based on a large stock portfolio with an equal exposure to 45 different risk factors, viz. the returns on the 45 country indices within the MSCI All Country World Index. Overall, this index includes over 8,500 securities, and we consider the perspective of investors that are fully-diversified in each local market. In fact, trading on the indices themselves is also possible, on some indices via exchange traded funds and on all indices via the equity index swaps provided by Morgan Stanley. Assuming the equal portfolio weighting is held constant ensures that changes in VaR estimates over time only result from risk factor characteristics, rather than from portfolio re-constructions.

Our historical returns series begins in 29 Sept 1997 and ends on 11 Jan 2010, a total of 3206 observations. A time series of this portfolio’s historical returns is shown in Figure 1. Its cumulative returns follow the typical trend of a global portfolio exposed to the turbulent and tranquil market conditions of the last decade. The dot-com bubble, which burst in 2000, is clearly visible, while the recent 2008 banking crisis is marked by the high levels of volatility in the autumn of that year. A major challenge for VaR models will be capturing these periods of extreme returns without over-estimating risk during periods of stable growth.

Ultimately, our objective is to control the multivariate skewness and kurtosis of our
simulated samples, while using a variety of ROM simulations that have the same target multivariate moments but different moments in their marginal distributions. In Figure 2 we plot the multivariate skewness and kurtosis of daily returns on these indices over time, calculated using rolling windows of 500 observations each covering a period of about two years. Clearly, the two measures are closely related and it seems reasonable to suppose that they are driven by a common factor.

5.2. Calibration of $L$ Matrices

For the purposes of estimating a time series of portfolio VaR, ROM simulations will be based on a Type I deterministic $L$ matrix $L_{mn}^k$ of the form (10) with parameters $m$ and $k$ calibrated to the multivariate sample skewness $\hat{\tau}$ and kurtosis $\hat{\kappa}$ by numerically minimizing the objective function:

$$f_n(m, k; \hat{\tau}, \hat{\kappa}) = \sqrt{\left(\frac{\tau_M(L_{mn}^k)}{\hat{\tau}} - \hat{\tau}\right)^2 + \left(\frac{\kappa_M(L_{mn}^k)}{\hat{\kappa}} - \hat{\kappa}\right)^2}.$$  \hspace{1cm} (15)
Figure 2: Evolution of multivariate skewness and kurtosis for a set of 45 MSCI indices on a
daily rolling window with 500 observations.

This is an integer-based optimisation problem so derivative-free methods must be employed.\footnote{The \texttt{fminsearch} subroutine in MATLAB, which applies the Nelder-Mead simplex method, can be used to solve this problem. See Lagarias et al. [1998] for details.} Given the complexity of such optimisation algorithms we found it more efficient to calculate $\tau_M(L_{mn}^k)$ and $\kappa_M(L_{mn}^k)$ for a large range of integer pairs $(m, k)$ and form a grid of $f_n(m, k; \hat{\tau}, \hat{\kappa})$. An example of such a grid is depicted in Figure 3, based on targets $\hat{\tau} = 1386$ and $\hat{\kappa} = 4111$ calculated from a two-year sample on MSCI daily returns ending on 18 Sept 2008, when news of the Lehman Brothers collapse reached the markets.

The surface of (15) values is very smooth because there is no simulation error in deterministic ROM simulations. The minimum point on the grid occurs at the point $(183,35)$ where the objective function takes the value 0.0076. With these parameters, our ROM simulated samples have a skewness of 1390 and a kurtosis equal to 4141, which are reasonably close to the historical targets. Since a simulation with the above parameters will only have 183 observations, ROM simulations based on the calibrated $L$ matrix are repeated many times and then combined using sample concatenation. Ledermann et al. [2011] proves that multivariate kurtosis is unchanged under sample concatenation, but multivariate skewness may decrease marginally as the number of concatenations increases. However, Figure 2 indicates that Mardia skewness and kurtosis are governed by a common market driver, so that simulations targeting either one of these moments should capture the relevant characteristics. Moreover, in our stress tests we shall control the skewness and kurtosis of the individual risk factor simulations by choosing three different types of random orthogonal matrices in the ROM VaR computations, plus a random rotation matrix to ensure that the simulated
marginals have negative skew, so that extreme negative returns on the MSCI county index risk factors are simulated more frequently than extreme positive returns.

5.3. VaR Models and Backtests

We now test the performance of three variants of ROM VaR, also choosing some standard unconditional VaR models as benchmarks. Every time the VaR is estimated, 10,000 scenarios on daily returns to the 45 MSCI country indices are generated using ROM simulations based on the $L$ matrix of Type 1 with parameters calibrated as described above, and with either a random upper Hessenberg, random Cayley or random exponential rotation matrix $R_n$ in (7). For two of our benchmarks we also compute 10,000 MC simulations assuming a multivariate normal distribution, and then assuming a multivariate Student-t distribution with 5 degrees of freedom, for the risk factor returns. We also report results based on an exact analytic solution, which is available only in the multivariate normal case.

Starting on 27/08/1999 we use the previous 500 observations (approx. 2 years) of data to calculate the target means, covariances and higher multivariate sample moments for our simulations. These sample moments are used to simulate 1-day ahead portfolio returns (but only the mean and covariance matrix are used for the MC simulations) and from these we estimate daily VaR at 1% significance. As mentioned in the introduction, this VaR parameter choice is consistent with Basel II banking recommendations. Our third benchmark model is the historical VaR, with estimates derived from this same sample of 500 observed daily returns. Then, having calculated VaR estimates on 27/08/1999, we roll the in-sample period
forward by one day and re-estimate the VaR according to ROM, MC or historical models, continuing this until all data are exhausted.

<table>
<thead>
<tr>
<th>Model</th>
<th>% Exceedances</th>
<th>Unconditional</th>
<th>Independence</th>
<th>Conditional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal (Analytic)</td>
<td>2.73%</td>
<td>55.84</td>
<td>29.99</td>
<td>85.82</td>
</tr>
<tr>
<td>Normal (MC)</td>
<td>2.70%</td>
<td>53.80</td>
<td>30.67</td>
<td>84.48</td>
</tr>
<tr>
<td>Student-t MC</td>
<td>1.52%</td>
<td>6.27</td>
<td>17.97</td>
<td>24.23</td>
</tr>
<tr>
<td>Historical</td>
<td>1.37%</td>
<td>3.31</td>
<td>15.08</td>
<td>18.39</td>
</tr>
<tr>
<td>Hessenberg ROM</td>
<td>0.89%</td>
<td>0.36</td>
<td>5.66</td>
<td>6.02</td>
</tr>
<tr>
<td>Cayley ROM</td>
<td>1.18%</td>
<td>0.86</td>
<td>3.59</td>
<td>4.45</td>
</tr>
<tr>
<td>Exponential ROM</td>
<td>2.07%</td>
<td>23.89</td>
<td>23.59</td>
<td>47.49</td>
</tr>
</tbody>
</table>

Table 1: Percentage of days when loss exceeded the VaR forecast and coverage test results for 1% daily VaR. The 1% critical values are 6.63 for the unconditional and independence tests and 9.21 for the conditional test.

Finally, to test the specification of each VaR model we apply the coverage tests of Christoffersen [1998]. A VaR model passes a coverage test if the relevant test statistic is below the 1% critical value of the appropriate chi-squared distribution. The results are summarized in Table 1. They indicate that only the Hessenberg and Cayley ROM models pass all three coverage tests. Three models (exponential ROM and analytic/MC normal) even fail the unconditional test, because they do not take account of the highly leptokurtic nature of the portfolio’s daily returns, so their corresponding VaR estimates are too low. All models except the Hessenberg and Cayley ROM models fail the independence and conditional coverage tests because they are unable to capture the clustering in exceedances, especially around the time of the banking crisis.

5.4. Sample Bias in Duffie-Pan Methodology

For the purpose of examining sample risk in the D-P methodology, we now calculate a stressed covariance matrix using historical returns on the MSCI index risk factors between 01 Jan 2006 and 31 Oct 2008 (739 days), to capture the extreme risk factor volatility and co-dependency experienced during the lead up to the 2007 credit crunch and the onset of the 2008 global financial crisis. Using this stressed covariance matrix, we construct risk factor scenarios using (11) and with a data-specific $L$ matrix representing the entire historical sample. The result is thus a stressed VaR for our equally-weighted portfolio on 11 Jan 2010.

Based on the entire series of daily returns the 1-day 1% historical VaR of the portfolio was 3.30% of the portfolio value on 11 Jan 2010. With the identity matrix in place of the
random orthogonal matrix in (11) we obtain the D-P daily sVaR estimate, which is 4.29% of the portfolio value, so in this case the sVaR/VaR ratio is $4.29/3.30 = 1.3$. We then calculate further sVaR estimates using different random orthogonal matrices in (11). For each random orthogonal matrix type we generate 1000 sVaR/VaR ratios, where the sVaR estimate is ROM simulated and the VaR estimate is calculated historically as above.

<table>
<thead>
<tr>
<th>Stress Levels</th>
<th>Mean</th>
<th>Stdev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hessenberg</td>
<td>1.1527</td>
<td>0.0620</td>
<td>0.9600</td>
<td>1.3504</td>
</tr>
<tr>
<td>Cayley</td>
<td>1.0906</td>
<td>0.0420</td>
<td>0.9668</td>
<td>1.2369</td>
</tr>
<tr>
<td>Exponential</td>
<td>1.1127</td>
<td>0.0408</td>
<td>0.9589</td>
<td>1.2706</td>
</tr>
<tr>
<td>Identity</td>
<td>1.3000</td>
<td>0.0000</td>
<td>1.3000</td>
<td>1.3000</td>
</tr>
</tbody>
</table>

Table 2: Statistics based on 1000 sVaR/VaR ratios. Both sVaR and VaR are 1% daily measures. The sVaR estimates are ROM simulated while unstressed VaR is historical (3.30%).

The mean, standard deviation, minimum and maximum of these ratios are reported in Table 2. The results show that different stressed samples, although attributed to the same means, covariances and multivariate skewness and kurtosis, can give rise to a variety of sVaR estimates. Some Hessenberg simulated samples produce sVaR levels which are up to 1.35 times greater than unstressed VaR, but on average, the ROM simulated sVaR estimates are lower than their D-P counterparts. The variability in our sVaR estimates is not too surprising since these calculations are based on 1% quantiles. Such estimates are sensitive to a handful of scenarios lying in the lower tail of the simulated portfolio returns distribution and the tails of a ROM simulation can vary considerably.

5.5. Stressing Higher Moments

Recall that a limitation of the D-P method is that samples are only ‘stressed’ through their covariances. We now apply the higher moment sVaR technique described in Section 4 to our portfolio with MSCI index risk factors. Over the total sample (29/09/1997 to 11/01/2010) the risk factor returns have a multivariate kurtosis of $\kappa = 4046$, but in our stressed sample (01/01/2006 to 31/10/2008) the kurtosis is $\tilde{\kappa} = 4487$, i.e. about 11% greater than the unstressed kurtosis. Because the Mardia [1970] measures are driven by a common factor the multivariate skewness is also about 11% greater during the stressed sample. Using these stressed parameters we will calculate 0.1% and 1% sVaR estimates.

Targeting $\tilde{\kappa}$ we first set $r = 1$ in (13) and solve for $p$, which we find to be 236. We then form a stressed sample of the form (12) using a random orthogonal matrix (Hessenberg,
Cayley or exponential) and a random sign matrix. These sign matrices affect the marginal skewness of our stressed samples, switching positive to negative skewness in a risk factor’s marginal density, and vice versa. For this application the parameters in the random sign matrices are chosen to induce a negative skew, which is an important feature to include in sVaR estimates of equity portfolios.

We also calculate sVaR estimates by fixing \( r = 5 \) and \( r = 10 \) in construction (12). For each value of \( r \), and for each class of orthogonal matrix, we estimate sVaR 1000 times. The average value of these sVaR estimates, divided by the appropriate 0.1% or 1% unstressed historical VaR estimate, are reported in Table 3. The rows labelled ‘L matrix length \( p \)’ is the closest integer to the quadratic root defined by (14). The first column corresponds to the D-P sVaR/VaR ratio, with stressed covariances but unstressed kurtosis. The columns with \( r \geq 1 \) correspond to the additional higher moment stress tests.

<table>
<thead>
<tr>
<th></th>
<th>0.1% Stressed VaR/Unstressed VaR Ratios</th>
<th>1% Stressed VaR/Unstressed VaR Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r = 0 )</td>
<td>( r = 1 )</td>
</tr>
<tr>
<td>Hessenberg</td>
<td>1.3562</td>
<td>1.3875</td>
</tr>
<tr>
<td>Cayley</td>
<td>1.3562</td>
<td>1.4232</td>
</tr>
<tr>
<td>Exponential</td>
<td>1.3562</td>
<td>1.3562</td>
</tr>
<tr>
<td>L matrix length ( p )</td>
<td>( n/a )</td>
<td>236</td>
</tr>
</tbody>
</table>

Table 3: Targeting a multivariate kurtosis of \( \tilde{\kappa} = 4487 \) with different numbers of augmentations \( r \). The averages of 1000 sVaR/VaR ratios are reported. The 0.1% and 1% unstressed VaR estimates are 6.71% and 3.30% of the portfolio value respectively.

In general, the more ROM simulations we include in the overall sample the higher the sVaR estimate, because the number of observations reflecting a stressed scenario increases relative to the number of historical observations (3206 days). For example, when \( r = 1 \) we have 236 observations generated from distributions with extremely high kurtosis and when
$r = 10$ we have 1270 observations reflecting a stressed but still lower level of kurtosis, but in all cases the overall sample has the target kurtosis $\tilde{\kappa} = 4487$.

At the 0.1% significance level, using stressed Hessenberg and Cayley ROM simulations to augment the historical sample will increase the sVaR estimates, while using exponential ROM simulations will actually decrease these levels. This is because different orthogonal matrices produce different marginal density shapes, even though all simulations target the same multivariate kurtosis (see Ledermann and Alexander [2012]). The marginals densities generated by exponential ROM simulations tend to be symmetric and mesokurtic so these simulations rarely contribute any extreme scenarios to the already leptokurtic unstressed historical sample. Hence, the overall proportion of large negative tail scenarios in the combined stressed sample actually decreases.

Hessenberg and Cayley simulated samples are typically leptokurtic and negatively skewed. Sometimes, however, the shape of a distribution is such that adding probability mass to the 0.1% tail shifts the 1% quantiles to the right. This phenomenon is a common feature of some leptokurtic distributions and may explain why a few 1% sVaR estimates actually decrease when historical data is augmented with certain ROM simulations.

5.6. Hessenberg vs Cayley Rotation Matrices

Given the known, mesokurtic characteristics of the marginal densities obtained from exponential ROM simulations only upper Hessenberg or Cayley ROM simulations are suitable for stressed VaR applications. To decide which type of rotation matrix is better we ask: ‘how do ROM sVaR estimates change as the general level of kurtosis is increased by some multiplicative factor?’ Is there a monotonic relationship between the sVaR and multivariate kurtosis for both types of ROM simulation? To answer this question, first recall that the kurtosis targeted in Table 3 was 11% more than the unstressed kurtosis. Now fixing $r = 5$, we estimate sVaR assuming a 5%, 10% and 25% increase in unstressed kurtosis. Average results, relative to unstressed VaR estimates, are given in Table 4.

With $c = 1$ the values in the first column of Table 4 should be compared with the D-P 0.1% and 1% sVaR/VaR ratios, which are 1.3562 and 1.3000 respectively. Increasing $c$ has the effect of increasing the multivariate kurtosis relative to the unstressed sample. At the 0.1% significance level both Hessenberg and Cayley ROM simulations yield an sVaR/VaR ratio that increases with the general level of kurtosis, but at the 1% level only the Cayley ROM simulations produce sVaR/VaR ratio that always increases with the unstressed level of multivariate kurtosis.

Ledermann and Alexander [2012] show that, when based on Type I $L$ matrices, the marginal densities of Hessenberg and Cayley ROM simulations have similar levels of positive
Table 4: Targeting multivariate kurtosis factor increases $c$ with $r = 5$ augmentations for upper Hessenberg and Cayley ROM simulations. The averages of 1000 sVaR/VaR ratios are reported. The 0.1% and 1% unstressed VaR estimates are 6.71% and 3.30% of the portfolio value respectively.

excess kurtosis, but they differ markedly in their skewness. In particular, Cayley rotation matrices produce a very high positive skew in the marginals while Hessenberg matrices yield a smaller, but still significant negative skew. Since random rotation matrices can be applied in the ROM simulation (just as we have done in our stress tests) to ensure a negative skew in the marginals, we recommend that, rather than using upper Hessenberg matrices, users of ROM simulation should apply random rotation matrices in combination with random Cayley matrices. This way the sVaR/VaR ratio is more likely to increase with the general level of kurtosis in the unstressed sample, plus the marginal risk-factor distributions will have a more significant negative skew.

6. Summary and Conclusions

Conditional models are well-known to be more accurate for VaR estimation than standard unconditional models such as historical or MC simulation. But the application of conditional models, such as GARCH, to computing VaR estimates for risk capital calculation poses too many problems in practice to be used by most financial institutions. Indeed, in a recent survey 95% of responding banks used either historical simulation or standard MC simulation for computing VaR. On the other hand, many papers that examine the accuracy of VaR models find that these VaR models fail even the simple backtests that regulators recommend.
Our empirical study also found that this was the case with standard historical VaR and VaR based on a multivariate normal assumption for risk factor returns. The Student-t MC VaR just passed the simple (unconditional coverage) test but failed the independence and conditional coverage tests. However, two of the three ROM VaR models that we tested passed all three specification tests. These were generated by using an upper Hessenberg matrix, or a Cayley matrix, in the ROM simulation algorithm.

In response to the financial crisis the Basel committee on Banking Supervision has recently recommended that all banks augment their market risk capital to add on a reserve that is directly related to the stressed Value-at-Risk (sVaR) of their portfolios. This risk measure should correspond to the VaR calculated during a typical period of significant market stress. Since the large majority of banks employ historical simulation as their VaR method of choice, the industry is in need of a method to compute stressed VaR based on this approach. The challenge they face is to obtain sufficient data that reflects the stressed scenario in all risk factors, otherwise the precision of the stressed VaR estimates will be compromised by the measurement of an extreme quantile based on a small sample.

Duffie and Pan [1997] introduced a method to generate more precise sVaR estimates by transforming a larger historical sample in such a way that it reflects a stressed risk factor covariance matrix. However, their approach utilises only one historical sample. We have applied ROM simulation with data-specific $L$ matrices to generate as much stress testing data as we like. As in the standard ROM VaR algorithm, randomness is introduced to each simulation via an appropriate choice of rotation matrix, and each sample on the risk factors will have exactly the target mean and covariance matrix. We used ROM stress tests to highlight the sample risk associated with the D-P approach, by showing how different stressed historical samples with identical first four multivariate moments can lead to quite different sVaR estimates.

The D-P approach only stresses the risk factor volatilities and correlations, not their higher moments. We have specified an algorithm for ROM simulated stress testing based on stressed levels of multivariate moments, in addition to or instead of stressing the covariance matrix. As both skewness and kurtosis are known to increase markedly during crises periods this is another welcome extension of the D-P methodology, especially since most banks use historical simulation for VaR estimation. We recommend that users of ROM VaR employ upper Hessenberg or Cayley rotation matrices, with the latter being more appropriate for stressed VaR computations. Random reflection matrices may also be employed to induce a high negative skewness in the marginal risk factor returns.
References


