Average price portfolio insurance as optimal implementation of life-cycle investment strategies

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ABSTRACT

We design average price portfolio insurance (APPI) strategies with an investment floor and a buffer that is a power of a generalised geometric average of the underlying asset price. We prove that APPI strategies are optimal for investors with hyperbolic absolute risk aversion who become progressively more risk averse over time. During the averaging period, APPI strategies reduce the proportion of wealth allocated to the risky asset, which is the traditional life-cycle investment recommendation. We compare the sensitivities of the fair price of equivalent payoffs generated by average and constant proportion portfolio insurance strategies and illustrate the performance of APPI strategies.

JEL Classification Codes: G11, G12, G13, G17

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1. INTRODUCTION

Most investors are risk averse and require investment strategies which limit their exposure to risk. Merton (1971) and Brennan and Solanki (1981) show that the optimal payoff for an investor with an hyperbolic absolute risk aversion (HARA) utility function in an economy with a risk-free and a log-normally distributed risky asset consists of an investment floor plus a power function of the risky asset price. Perold (1986) and Black and Jones (1987) introduce the constant proportion portfolio insurance (CPPI) strategy as a replication strategy in continuous time for this optimal payoff. A CPPI strategy and the corresponding payoff are specified through two parameters: a floor and a multiplier. The investment floor, growing at the risk-free rate, splits the portfolio into a guaranteed part and a buffer, the excess over the floor. A constant multiple of the buffer is invested in the risky asset and the rest in the risk-free asset. The greater the floor and the lower the multiplier the lower the exposure to the risky asset and vice versa. The two parameters of the HARA utility function determine the optimal values of the floor and the multiplier.

The most common alternative portfolio insurance strategy to CPPI is the standard option based portfolio insurance (OBPI) strategy, i.e., a bond plus the replication of a standard call option (or the underlying risky asset plus the replication of a put option). In effect, a CPPI strategy is also an option based portfolio insurance, if one considers the power payoff as a power option. The payoffs of both strategies are, in theory, path independent in the sense that their values at any point in time depend solely on the price of the underlying asset at that time, but unlike a standard call, which is defined for a chosen time horizon, the payoff of a CPPI strategy is defined over any maturity.¹

Portfolio insurance strategies suit long-term investors who may, for example, be saving for their retirement. They guarantee a minimum return, yet, whether CPPI or OBPI, the value of the protected portfolio is highly uncertain on the upside as it depends solely on the price of the underlying risky asset. So, the return to a pensioner buying an annuity over the last few years would have varied widely depending on its exact retirement date. Average price (Asian) options were introduced to decrease an investor’s dependency on the final price of the risky asset by defining their payoffs as function of an average price before maturity. The first study on pricing standard (call and put) Asian options is Kemna and Vorst (1990), who find that there is no analytic pricing formula for Asian options on arithmetic means of log-normal processes. They obtain prices using Monte Carlo simulations and find that

¹Studies comparing CPPI and OBPI strategies are e.g. Bertrand and Prigent (2005), Annaert et al. (2009) and Zagst and Kraus (2011).
arithmetic Asian call options are always cheaper than the corresponding standard European call options. Pricing options on a geometric average of log-normal prices, on the other hand, yields simple analytical results. Kemna and Vorst (1990) and Turnbull and Wakeman (1991) evaluate geometric average price call options and find that geometric average price call options are lower bounds for the prices of arithmetic average price options. The latter are more popular because more easily understood by investors. A vast stream of literature follows on approximations for pricing options on arithmetic averages. We refer the reader to the following studies: Carverhill and Clewlow (1990) use fast Fourier transforms, Turnbull and Wakeman (1991) approximate the average by fitting integer moments, Geman and Yor (1993) apply a Laplace transform approach, Bouaziza et al. (1994) derive a valuation formula using a slight linear approximation that has a formal upper bound for the approximation error and more recently Fusai and Meucci (2007) price discretely monitored geometric and arithmetic Asian options modelling the underlying with a generic Lévy process.

We extend the concept of average price standard options to power options and therefore to average price portfolio insurance (APPI) strategies. We define a generalised geometric average of the underlying risky asset price over a certain maturity. Using the concept of a standard CPPI, the payoff of an APPI product is the sum of an investment floor plus a power of this generalised geometric average. The payoff of an APPI product is therefore path dependent and must be defined over a chosen maturity. Investors do not risk losing a large fraction of their upside return just before maturity as the decreasing dependency on the current price decreases the volatility of the portfolio as it approaches maturity. The decreasing portfolio exposure to the risky asset is in line with classic life-cycle investment strategies: APPI strategies suit investors who become progressively more risk averse over time.

Figure 1 shows the relationship between the choice of utility parameters and the optimal payoff function: Investors can either define their utility function which then sets their optimal payoff function and a corresponding replication strategy with the underlying risky asset, or they can start in the opposite direction and find their implied utility parameters. There are one to one relationships between utility parameters, optimal payoff function of a generalised geometric average (defining index) and replication strategy.

Some other alternatives to standard CPPI products have already been proposed. Boulier and Kanniganti (1995) introduce variable floor CPPI strategies that adjust the floor should the exposure to the risky asset become too low or too high. Chen and Chang (2005) make the multiplier variable to adjust to changing market environments. Our extension of the CPPI
Figure 1: Correspondences between a HARA utility function, an optimal investment payoff function, its defining index and its replication strategy with the underlying asset spectrum suits investors who want to decrease their risk over time in a manner consistent with a specified increase in risk aversion. The literature has arguments for and against this life cycle investment recommendation. Bodie et al. (1992) stress that the optimal risky asset mix over a lifetime should be constant under simplistic assumptions (namely, i.i.d. returns on investment, independent from labour income and consumption, and time-independent utilities). On the other hand, studies by Jagannathan and Kocherlakota (1996) and Cocco et al. (2005) argue that labour income (or human capital) should be introduced as an additional component in the risky asset mix that can offset investment risks but which decreases over time. Consequently, individuals should take more risky investments when they are young and move towards less risky investments as they grow older and their capacity to earn decreases. Progressive habit formation leading to greater risk aversion has also been proposed as another argument for reducing investment risks over time.

The outline of the paper is as follows. Section 2 describes our market model, a standard Black–Scholes economy, and defines the averaging process. We construct an APPI strategy that replicates a power payoff on a standard and a generalised geometric average in Section 3 and derive the sensitivities of the fair price of the APPI strategy to changes in the underlying risky asset price and volatility in Section 4. In Section 5 we show that an APPI strategy is optimal for investors with HARA utilities with risk tolerance decreasing over time. We illustrate the performance of several APPI strategies in Section 6 and conclude in Section 7.
2. Market model and generalised geometric average process

We assume a Black–Scholes economy, that is, a complete financial market in which investors can freely trade in a risk-free asset and a risky asset; the risk-free asset grows at a constant rate \( r_f \) and the risky asset price follows a geometric Brownian motion (GBM)

\[
S(t) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_S(t) \right),
\]

where \( S(0) \) is the initial asset price, the constants \( \mu \) and \( \sigma \) denote the continuous trend and standard deviation of the process, respectively, and \( W_S(t) \) is a Brownian process. There are no trading restrictions nor transaction costs. The standard CPPI strategy dynamically rebalances in continuous time a portfolio invested in the risk-free and the risky asset so as to maintain the exposure to the risky asset at a constant multiple of the buffer, the excess value of the portfolio above a floor that grows at the risk-free rate. The payoff can be found by applying Itô’s Lemma to the strategy dynamics. Perold and Sharpe (1988) derive the CPPI payoff \( CPPI(t; F, m) \) for any \( t > 0 \) as

\[
CPPI(t; F, m) = Fe^{r_f t} + B_p(t)
= Fe^{r_f t} + (w(0) - F) \left( \frac{S(t)}{S(0)} \right)^m \exp \left( (1 - m) \left( r_f + \frac{1}{2} \sigma^2 \right) t \right),
\]

where \( B_p(t) \) denotes the buffer, \( w(0) \) the initial wealth, \( F \) the initial investment floor and \( m \) the multiplier. Thus the payoff of a standard CPPI is a power function of the underlying. A CPPI strategy is therefore the replication strategy of a power option plus a floor. As with a standard option, the CPPI payoff is independent of the underlying asset price path. But, unlike a standard option with the payoff defined at maturity, a CPPI strategy power payoff can be extended to any maturity; a CPPI strategy is open ended. The multiplier controls the curvature of the strategy payoff: A multiplier greater than 1 gives a convex payoff and a multiplier smaller than 1 gives a concave payoff (see Bertrand and Prigent, 2005). The choice of the floor affects only the volatility of the strategy but the multiplier influences all moments of the payoff distribution.

To construct an APPI strategy we define an average price and use it as the underlying index. We consider an averaging period from time \( t_a \) to maturity \( T \), with \( 0 \leq t_a \leq T \). The two most commonly used types of averages are the arithmetic and the geometric averages. The
arithmetic average sums up prices over the averaging period:

\[ A(t_a, T) = \frac{1}{\tau} \int_{t_a}^{T} S(t)dt, \]  

(3)

where \( \tau = T - t_a \). The logarithm of the geometric average sums up the log-prices over the averaging period:

\[ G(t_a, T) = \exp \left( \frac{1}{\tau} \int_{t_a}^{T} \ln(S(t))dt \right). \]  

(4)

In the following we refer to the value at time \( t \) of this standard geometric average as \( G(t_a, t) \) for any \( 0 \leq t \leq T \). The difference between (3) and (4) may be small but the geometric average is always lower than the arithmetic average for any volatile price (see Beckenbach and Bellman, 1961). Like Angus (1999) we argue that financially there is little effect whether we choose the arithmetic or the geometric mean when defining Asian derivatives. The geometric mean of a GBM process can easily be expressed analytically; the arithmetic mean requires numerical approximations but may still be easier to understand by investors.

To generalise the concept of averaging we introduce \( \alpha(t) \) as a decreasing weighing function for the underlying price at time \( t \) with \( \alpha(t) \geq 0 \) and \( \alpha(0) = 1 \) and define a generalised geometric average as the Riemann–Stieltjes integral

\[ G(\alpha, t_a, T) = \exp \left( \int_{t_a}^{T} \ln(S(t))d\alpha(t) \right), \]  

(5)

where \( d\alpha(t) \) denotes the variation of \( \alpha(t) \) over \( dt \). This integral exists as \( \ln(S(t)) \) is continuous and \( \alpha(t) \) is bounded. A generalised average gives investors greater flexibility to match the time variation of their risk aversion in the design of their optimal payoff function. For some choices of \( \alpha(t) \) we can find closed-form solutions for (5). One instance is \( \alpha(t) = \frac{T-t}{\tau} \) over \( t_a \leq t \leq T \) and hence \(-d\alpha(t) = -\alpha'(t)dt = \frac{dt}{\tau} \) yielding the standard definition of the geometric average as in (4). We refer to this weighing function as \( \bar{\alpha}(t) \).

Kemna and Vorst (1990) and Turnbull and Wakeman (1991) derive the geometric average \( G(t_a, T) \) of a GBM process (1) over the period \([t_a, T]\) as a function of \( S(t_a) \) and obtain

\[ G(t_a, T) = S(t_a) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{\tau}{2} + \sigma W \left( \frac{\tau}{3} \right) \right), \]  

(6)

where \( W(\tau) \) is a Brownian process correlated with \( W_S(\tau) \) but distinct from it. Compared
to the underlying price process, the geometric average has half the drift and a third of the variance. Consequently, in a risk-neutral world with $\mu = r^f$ its discounted value is not a martingale; the geometric average is not a tradeable asset. To examine APPI strategies starting before the averaging period, we include the stock prices before $t_a$ and derive the price of the standard geometric average at $T$ as a function of $S(0)$.

**Proposition 2.1.** The time $T$ value of the standard geometric average process over interval $[t_a, T]$ as a function of the initial price $S(0)$, $0 \leq t_a \leq T$ is

$$G(t_a, T) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) (t_a + \frac{T}{2}) + \sigma W \left( t_a + \frac{T}{3} \right) \right)$$

(7)

and for $t_a < t < T$ it is

$$G(t_a, t) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t_a + \sigma W(t_a) \right) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{t}{2} + \sigma W \left( \frac{T}{3} \right) \right),$$

(8)

where $\bar{\alpha}(t) = \frac{T - t}{\tau}$.

**Proof.** With (1) and (6) we get the geometric average process as a function of $S(0)$ at time $T$ as

$$G(t_a, T) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t_a + \sigma W(t_a) \right) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{T}{2} + \sigma W \left( \frac{T}{3} \right) \right),$$

which gives (7) when we merge the exponential terms.

For any time $t$ with $t_a < t < T$ the geometric average as a function of $S(t_a)$ is

$$G(t_a, t) = S(t_a) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{t}{2} (1 - \bar{\alpha}(t))^2 \right) + \sigma W \left( \frac{T}{3} (1 - \bar{\alpha}(t))^3 \right).$$

Hence the value of the geometric average as a function of $S(0)$ at $t$ is given by (8) as we write $S(t_a)$ as a function of $S(0)$. For $t \to T$ (8) yields (7).

The closed form of the geometric average price process is obviously equal to the geometric Brownian price process for $t \leq t_a$. So we can summarise the geometric average process over
As:
\[
G(t_a, t) = \begin{cases} 
S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_s(t) \right) & \text{if } 0 \leq t \leq t_a \\
S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) (t_a + \frac{5}{3}(1 - \alpha(t))^2) + \sigma W \left( t_a + \frac{5}{3}(1 - \alpha(t))^3 \right) \right) & \text{if } t_a < t < T \\
S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) (t_a + \frac{2}{3}) + \sigma W \left( t_a + \frac{2}{3} \right) \right) & \text{if } t = T.
\end{cases}
\]

3. Construction of an APPI Strategy

The standard CPPI strategy leads to the power payoff (2). Similarly, we define an APPI strategy as the dynamic asset allocation strategy in continuous time replicating the corresponding payoff based on the generalised geometric average price $G_{\alpha}$ over the time interval $[t_a, T]$, that is, at maturity $T$:

\[
A_{\alpha}(T; F, m, t_a, T) = Fe^{r' T} + B_{\alpha}(T) = Fe^{r' T} + (w(0) - F) k_{\alpha} \left( \frac{G_{\alpha}(t_a, T)}{S(0)} \right)^m.
\]

The buffer $B_{\alpha}(T)$ is now a power function of the generalised geometric average (5) and $k_{\alpha}$ is a normalisation factor such that the risk-neutral price of the APPI payoff under the risk-neutral probability $Q$ is equal to initial wealth $w(0)$. With the filtration $F_0$ at $t = 0$ this yields

\[
A(0; F, m, t_a, T) = \mathbb{E}_Q \left[ A(T; F, m, t_a, T) e^{-r' T} | F_0 \right] = w(0)
\]

and hence

\[
k_{\alpha} = \mathbb{E}_Q \left[ \left( \frac{G_{\alpha}(t_a, T)}{S(0)} \right)^m e^{-r' T} | F_0 \right]^{-1}.
\]

For the special case of the standard geometric average with $G(t_a, T)$ the normalisation factor has a closed-form solution which yields

\[
k_{\bar{\alpha}} = \exp \left( -m \left( r' - \frac{1}{2} \sigma^2 \right) \left( T - \frac{1}{2} \tau \right) - \frac{1}{2} m^2 \sigma^2 \left( T - \frac{2}{3} \tau \right) + r' T \right).
\]

As $k_{\alpha}$ depends on the maturity $T$ we have to fix the maturity of the APPI payoff (9) to calculate its fair price, $A_{\alpha}(t)$ at any time $t$, $0 \leq t \leq T$ and lose the open-endedness of a
standard CPPI strategy. We need to do this to find out how to replicate the APPI payoff. If \( 0 \leq t \leq t_a \) the fair price depends only on \( S(t) \) since the averaging has not started yet and the fair price is path independent. As soon as we enter the averaging period, i.e. \( t_a < t \leq T \), the fair price depends also on the realised average between \( t_a \) and \( t \): In \( [0, t_a] \) \( \mathcal{F}_t \) contains the knowledge of \( S(t) \) only and in \( (t_a, T] \) \( \mathcal{F}_t \) contains the knowledge of \( S(t) \) and \( G(t_a, t) \). Hence we consider the two time intervals \( [0, t_a] \) and \( (t_a, T] \) separately to price the APPI payoff.

**Theorem 3.1.** When \( 0 \leq t \leq t_a \) the risk-neutral price of the APPI payoff (9) is

\[
A_{\alpha}(t) = F e^{r_f t} + (w(0) - F) k_{\alpha} \left( \frac{S(t)}{S(0)} \right)^m E_Q \left[ \left( \frac{G_{\alpha}(t_a, T)}{S(t)} \right)^m S(t) \right] e^{-r_f(T-t)}, \tag{11}
\]

and for \( \bar{\alpha}(t) \) it yields

\[
A_{\bar{\alpha}}(t) = F e^{r_f t} + (w(0) - F) k_{\bar{\alpha}} \left( \frac{S(t)}{S(0)} \right)^m \\
\exp \left( m \left( r_f - \frac{1}{2} \sigma^2 \right) \left( T - t - \frac{1}{2} T \right) + \frac{1}{2} m^2 \sigma^2 \left( T - t - \frac{2}{3} T \right) - r_f(T - t) \right). \tag{12}
\]

When \( t_a < t \leq T \) the risk-neutral price of the APPI payoff (9) is

\[
A_{\alpha}(t) = F e^{r_f t} + (w(0) - F) k_{\alpha} \left( \frac{G_{\alpha}(t_a, t)^{1-\alpha(t)} S(t)^{\alpha(t)}}{S(0)} \right)^m E_Q \left[ \left( \frac{G_{\alpha}(t_a, T)}{S(t)} \right)^{\alpha(t)m} S(t) \right] e^{-r_f(T-t)}, \tag{13}
\]

and for \( \bar{\alpha}(t) \) it yields

\[
A_{\bar{\alpha}}(t) = F e^{r_f t} + (w(0) - F) k_{\bar{\alpha}} \left( \frac{G(t_a, t)^{1-\bar{\alpha}(t)} S(t)^{\bar{\alpha}(t)}}{S(0)} \right)^m \\
\exp \left( \bar{\alpha}(t)m \left( r_f - \frac{1}{2} \sigma^2 \right) \frac{T - t}{2} + \frac{1}{2} \bar{\alpha}(t)^2 m^2 \sigma^2 \frac{T - t}{3} - r_f(T - t) \right). \tag{14}
\]

**Proof.** We distinguish between Cases 1 and 2 as the time before and during the averaging period:

- **Case 1:** \( 0 \leq t \leq t_a \)

  Under the risk-neutral expectation on the generalised geometric average the price of
the APPI payoff at $t$ is

$$A_{\alpha}(t) = Fe^{r^f t} + (w(0) - F) k_{\alpha} \mathbb{E}_Q \left[ \left( \frac{G_{\alpha}(t, T)}{S(0)} \right)^{m} e^{-r^f(t-t)} \right| S(t) \right],$$

where the solvability of the risk-neutral expectation depends on the choice of $\alpha(t)$. For the standard geometric average $G(t_a, T)$ the expectation gives

$$\mathbb{E}_Q \left[ \left( \frac{G(t_a, T)}{S(t)} \right)^{m} \right| S(t) \right] =
\exp \left( m \left( r^f - \frac{1}{2} \sigma^2 \right) \left( T - t - \frac{1}{2} \tau \right) + \frac{1}{2} m^2 \sigma^2 \left( T - t - \frac{2}{3} \tau \right) \right)$$

and thus the risk-neutral price of (9) is equal to (12) for $0 \leq t \leq t_a$.

- **Case 2: $t_a < t \leq T$**

As we know the price process path until $t$ we split the geometric average $G_{\alpha}(t_a, T)$ into the known and unknown part and weigh them respectively:

$$G_{\alpha}(t_a, T) = G_{\alpha}(t_a, t)^{1-\alpha(t)}G_{\alpha}(t, T)^{\alpha(t)}.$$

For the APPI price we obtain

$$A_{\alpha}(t) = Fe^{r^f t} + (w(0) - F) k_{\alpha} G_{\alpha}(t_a, t)^{(1-\alpha(t))m} \mathbb{E}_Q \left[ \left( \frac{G_{\alpha}(t, T)}{S(0)} \right)^{\alpha(t)m} e^{-r^f(t-t)} \right| S(t) \right].$$

Evaluating the weighed expectation of the standard geometric average $G(t, T)^{\alpha m}$ gives

$$\mathbb{E}_Q \left[ \left( \frac{G(t, T)}{S(0)} \right)^{\tilde{\alpha}(t)m} \right| S(t) \right] =
\exp \left( \tilde{\alpha}(t)m \left( r^f - \frac{1}{2} \sigma^2 \right) \frac{T - t}{2} + \frac{1}{2} \tilde{\alpha}(t)^2 m^2 \sigma^2 \frac{T - t}{3} \right)$$

and thus the risk-neutral price of (9) is (14) for $t_a < t \leq T$.

The second terms in the sums in (12) and (14) are the values of the buffer $B_{\alpha}$ for $0 \leq t \leq t_a$ and $t_a < t \leq T$, respectively. \hfill \square

For the standard geometric average, taking into account $k_{\tilde{\alpha}}$ as in (10), at $0 \leq t \leq t_a$ the APPI payoff is a standard CPPI given by (2) and when $t_a < t \leq T$ we obtain the risk-neutral
price

\[ A_\alpha(t) = Fe^{r_f t} + (w(0) - F) \left( G(t_a, t)^{1-\bar{\alpha}(t)} \frac{S(t)^{\bar{\alpha}(t)}}{S(0)} \right)^m \]

\[ \exp \left( \frac{1}{2} m \left( r_f - \frac{1}{2} \sigma^2 \right) (\bar{\alpha}(t)^2 \tau - (2T - \tau)) + \frac{1}{6} m^2 \sigma^2 (\bar{\alpha}(t)^3 \tau - (3T - 2\tau)) + r_f t \right). \]

In the special case where averaging starts at \( t_a = 0 \) and therefore \( \tau = T \), which we call a Full APPI, we obtain

\[ A_\alpha(t) = Fe^{r_f t} + (w(0) - F) \left[ G(0, t)^{1-\bar{\alpha}(t)} \right]^{m} \]

\[ \exp \left( \frac{1}{2} m \left( r_f - \frac{1}{2} \sigma^2 \right) ((\bar{\alpha}(t)^2 - 1)T) + \frac{1}{6} m^2 \sigma^2 ((\bar{\alpha}(t)^3 - 1)T) + r_f t \right) \]

for all \( t \in [0, T] \).

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Figure 2: Effective APPI multiplier on the risky asset for the standard geometric average as a function of time for \( m = 2 \), \( T = 10 \) and three averaging periods \( \tau \)

Compared to the constant CPPI multiplier \( m \), we see that the effective APPI multiplier in (13), the exponent of \( S(t) \), is \( \alpha(t)m \) and decreases during the period of averaging with \( \alpha(t) \). With \( \bar{\alpha} \) the effective multiplier decreases linearly from \( m \) to 0. Thus APPI investors reduce their exposure to the risky asset over time and thereby the uncertainty in the final value of their portfolio. This is in line with the traditional life-cycle investment recommendation that investors should reduce progressively their exposure to risky assets as a long-term strategy, but it relates this exposure reduction to the parameters of their HARA utility function,
as we shall see in Section 5. In Figure 2 we show the effective APPI multiplier for the standard geometric average and three values of the averaging period \( \tau \). The multiplier starts decreasing when the averaging process begins. It decreases slowly for long periods of averaging and fast for shorter periods to reach zero at maturity. For any \( \alpha(t) \), the APPI price has the same curvature properties as the standard CPPI payoff: The payoff profile is convex when \( \alpha(t)m > 1 \), concave when \( \alpha(t)m < 1 \) and linear when \( \alpha(t)m = 1 \).

4. SENSITIVITIES

We derive the sensitivities of the fair price of a power payoff, or power option, with fixed maturity as we would for a standard option. We then compare the sensitivities to changes in the price and the price volatility of their common underlying risky asset for similar CPPI and APPI power options; by similar payoffs we mean payoffs that have the same maturity, floor and initial multiplier and that are calibrated to have the same initial fair price \( w(0) \).\(^2\)

For the APPI option the fair price is given by (11) and (13). The reference CPPI option payoff is given by (2) when \( t = T \), that is

\[
P(T; F, m, T) = Fe^{r_fT} + (w(0) - F) k_p \left( \frac{S(T)}{S(0)} \right)^m,
\]

where \( k_p \) is a normalisation constant such that the discounted payoff is equal to the initial wealth; therefore

\[
k_p = \mathbb{E}_Q \left[ \left( \frac{S(T)}{S(0)} \right)^m e^{-r_fT} | F_0 \right]^{-1} = \exp \left( (1 - m) \left( r_f + \frac{1}{2} m \sigma^2 \right) T \right).
\]

We now calculate the fair price of the CPPI option in (15) at any time before maturity.\(^3\)

**Proposition 4.1.** The risk-neutral price \( P(t; F, m, T) \) of the CPPI option with payoff (15)

\(^2\)Bertrand and Prigent (2005) examine the sensitivities of the payoff \( CPPI(t; F, m) \) to changes in the underlying price and changes in its volatility rather than the sensitivities of the fair price of a fixed-maturity option payoff.

\(^3\)The difference between a standard CPPI strategy which produces payoff (2) and the CPPI option is that the latter has a fixed maturity.
at time $t \in [0, T]$ is
\[
P(t; F, m, T) = Fe^{r' T} + B_p(T)
\]
\[
= Fe^{r' t} + (w(0) - F) k_p \left( \frac{S(t)}{S(0)} \right)^m \exp \left( (m - 1) \left( r' + \frac{1}{2} m \sigma^2 \right) (T - t) \right)
\] (17)

**Proof.** The risk-neutral price of the CPPI option at time $t$ is equal to
\[
P(t; F, m, T) = Fe^{r' t} + (w(0) - F) k_p \frac{1}{S(0)^m} \mathbb{E}_q[S(T)^m e^{-r'(T-t)}|S(t)]
\]
which gives (17) since
\[
\mathbb{E}_q[S(T)^m e^{-r'(T-t)}|S(t)] = S(t)^m \exp \left( (m - 1) \left( r' + \frac{1}{2} m \sigma^2 \right) (T - t) \right).
\]

With $k_p$ given by (16) we recover the CPPI payoff given by (2). For simplicity we call $P(t)$ the fair price of the CPPI option. We can now compare the sensitivities of the risk-neutral price of the CPPI option in (2) to the sensitivities of the risk-neutral price of the APPI option in (11) and (13).

The sensitivity of the CPPI option to changes in the underlying asset, the delta, of the power option is
\[
\Delta_p(t) = \frac{\partial P(t)}{\partial S(t)} = m \frac{1}{S(t)} B_p(t).
\]

The delta of the APPI option differs for times before and during the averaging and we obtain
\[
\Delta_a(t) = \frac{\partial A_a(t)}{\partial S(t)} = \left\{ \begin{array}{ll}
m \frac{1}{S(t)} B_a(t) & \text{if } 0 \leq t \leq t_a \\
\alpha(t) m \frac{1}{S(t)} B_a(t) & \text{if } t_a < t \leq T.
\end{array} \right.
\]

The deltas of both options are always positive, as $\alpha(t) \geq 0$, meaning that the greater the price of the underlying the greater the prices of these options. It is theoretically possible in a Black–Scholes economy to replicate the payoffs of the CPPI option in (17) and the APPI option as in (11) – (14) by maintaining in continuous time an exposure $\Delta_p S(t)$ and $\Delta_a S(t)$ to the risky asset, respectively. The delta of the APPI option decreases during the averaging period as the payoff becomes more heavily dependent on the average and less sensitive to
changes in the underlying. For the standard geometric average $\Delta_x$ reaches 0 at maturity.\(^4\)

The sensitivity of the delta to changes in the underlying price, the gamma, of the CPPI option is

$$\Gamma_x(t) = \frac{\partial \Delta_x(t)}{\partial S(t)} = m(m-1) \frac{1}{S(t)^2} B_x(t),$$

whereas the gamma of the APPI option is

$$\Gamma_a(t) = \frac{\partial \Delta_a(t)}{\partial S(t)} = \begin{cases} m(m-1) \frac{1}{S(t)^2} B_a(t) & \text{if } 0 \leq t \leq t_a \\ \alpha(t)m \ (\alpha(t)m-1) \frac{1}{S(t)^2} B_a(t) & \text{if } t_a < t \leq T. \end{cases}$$

For $m > 1$ the gamma of the APPI option turns from positive to negative during the averaging period.

Likewise the sensitivity to changes in the underlying’s volatility, or the vega of the CPPI option is\(^5\)

$$v_x(t) = \frac{\partial P(t)}{\partial \sigma} = \sigma m(m-1)(T-t)B_x(t),$$

whereas the vega of the APPI option is

$$v_a(t) = \frac{\partial A_a(t)}{\partial \sigma} = \begin{cases} \sigma m \left[ (m-1)(T-t) + \left( \frac{1}{2} - \frac{2}{3}m \right) \tau \right] B_a(t) & \text{if } 0 \leq t \leq t_a \\ \alpha(t) \sigma m \left( \alpha(t)mT - \frac{T-t}{2} \right) B_a(t) & \text{if } t_a < t \leq T. \end{cases}$$

Figure 3 illustrates the vegas of the CPPI option and the APPI option for the standard geometric average as a function of the underlying asset.\(^6\) The vega of the CPPI option is positive if $m > 1$ and negative if $m < 1$. The vega of the APPI option on the standard geometric average for $t \leq t_a$ is positive if

$$m > \frac{T-t - \frac{1}{2}\tau}{T-t - \frac{2}{3}\tau}.$$

During the averaging period the vega of the APPI option depends on the multiplier relative to $T$, $t$ and $\tau$. With $\tilde{\alpha}$ we find that $v_a$ is positive during the time of averaging when $m > \frac{3}{2\alpha}$ so it eventually turns negative before maturity (compare Figures 3(a) and (b) with (c) and

\(^4\)We need the delta of the APPI strategy relative to the risky asset price to replicate the payoff as the geometric average is not a tradeable asset.

\(^5\)The constants $k_x$ and $k_a$ do not impact the vegas as they are set at strategy initiation.

\(^6\)The market parameters used in Figure 3 are approximately the standard deviation of the S&P500 and the average rate on 13-weeks US Treasury Bills from 1990 to 2010.
Before averaging period

(a) $m = 2, t = 3$

Two years within averaging period

(b) $m = 2, t = 7$

(c) $m = 0.5, t = 3$

(d) $m = 0.5, t = 7$

Figure 3: Vegas of CPPI option $P$ and APPI option $A_\beta$ on a standard geometric average at time $t$ as a function of the risky asset price $S$ for $T = 10$, $\tau = 5$, $\sigma = 18\%$, $r_f = 3.5\%$, $F = 0.8$ and $G(5, 7) = 1$ (the horizontal axes show the value of the risky asset $S(t)$ at $t$ from 0.5 to 2 and the vertical axes the vegas)

(d)). For the chosen set of parameters the vega is positive at $t = 3$ if $m > 1.23$ and at $t = 7$ if $m > 2.50$.

Figure 4 illustrates the effects of the averaging process over time on the delta, gamma and vega of an APPI payoff on a standard geometric average. These sensitivities are expressed as a fraction of the buffer size, assuming the underlying asset remains constant. The delta decreases linearly during the averaging period. The gamma and the vega first decrease, become negative and then increase again. All three sensitivities approach 0 as the strategy gets closer to maturity.

5 Optimality of the APPI Payoff

The traditional argument to justify life-cycle investment strategies is the decrease of human capital over time. It is assumed that earnings generated by human capital are negatively correlated with investment performance, that is, if investment performance is poor, human
capital can compensate by generating more income. Vice versa, good investment performance can provide more leisure time. Alternatively, or in addition, one may assume an increasingly risk averse utility function over time. We use a general class of two-parameter utility functions, the HARA utilities, specified as follows:

\[
u(\tilde{w}) = \text{sgn}(\eta - 1) \left( 1 + \frac{\eta(t)}{\lambda(t)} (\tilde{w} - w(0)) \right)^{1 - \frac{1}{1+\eta}}
\]

where \( \tilde{w} \) denotes the present value of future wealth and \( w(0) \) is the investor’s initial wealth. Define as absolute risk tolerance (ART) the inverse of the Arrow Pratt absolute risk aversion (ARA) \( \gamma = -\frac{w''}{w'} \). With HARA utility function (18) the ART is

\[
\text{ART}(\tilde{w}) = -\frac{w'}{w''} = \lambda(t) + \eta(t)(\tilde{w} - w(0)),
\]

therefore ART varies linearly with wealth and ARA varies hyperbolically with wealth, hence the name HARA utility. With ART\((w(0)) = \lambda(t)\) the coefficient \( \lambda(t) \) is the initial absolute risk tolerance, and \( \eta(t) \) is the sensitivity of the ART to changes in wealth. For \( \eta \downarrow 0 \) a HARA
utility function converges to a negative exponential utility function, exhibiting constant ART \( \lambda(t) \) also called constant absolute risk aversion (CARA). For \( \eta \to 1 \) it converges towards a displaced logarithmic utility function. For \( \eta(t) = \lambda(t) \) the HARA utility function becomes a power utility function with constant relative risk aversion CRRA = \( \frac{1}{\lambda(t)w} \). Hence HARA utilities comprise traditional one-parameter utility functions as special cases.

Risk aversion imposes concavity of the utility function and therefore \( \lambda > 0 \) whereas constant or increasing ART with wealth imposes \( \eta \geq 0 \). With constant parameters \( \lambda \) and \( \eta \), Brennan and Solanki (1981) find that the optimal investment payoff when the underlying price is log-normally distributed with excess log-returns \( r \sim N \left( (\mu - r^f - \frac{1}{2}\sigma^2) T, \sigma^2 T \right) \) is

\[
w(T)e^{-r'T} = F^* + (w(0) - F^*) \exp \left( m^* r - m^* r^f T + \frac{1}{2} (1 - m^*) m^* \sigma^2 T \right), \tag{19}
\]

where \( m^* = \eta(t) \frac{\mu - r^f}{\sigma^2} \) is the optimal multiplier and \( F^* = 1 - \frac{\lambda(t)}{\eta(t)} \) is the optimal floor. This payoff is the CPPI payoff in (2).\(^7\)

Consider an investor with decreasing ART during period \((t_a, T)\) indicating increasing risk aversion. Specifically, let the ART decrease at the rate \( \alpha(t) \) by decreasing both parameters \( \lambda(t) \) and \( \eta(t) \) at the same rate, that is \( \lambda(t) = \alpha(t) \lambda_0 \) and \( \eta(t) = \alpha(t) \eta_0 \) with initial values \( \lambda_0 \) and \( \eta_0 \) at time \( t_a \). Then the optimal strategy parameters \( F^* \) and \( m^* \) in the optimal payoff function (19) change accordingly: The optimal multiplier is a function of \( \eta(t) \) and hence changes with time:

\[ m^*_\alpha(t) = \eta(t) \frac{\mu - r^f}{\sigma^2} = \alpha(t) m^*. \]

But as \( \eta(t) \) and \( \lambda(t) \) change at the same rate \( \alpha(t) \) their ratio \( \frac{\lambda(t)}{\eta(t)} \) stays constant and consequently the investor’s optimal floor remains constant at

\[ F^*_\alpha = w(0) - \frac{\lambda_0}{\eta_0}. \]

These are the fixed floor and the time varying multiplier leading to the APPI payoff \( A_\alpha \) as in (11)–(14) with the underlying \( G_\alpha \). That is, the function \( \alpha(t) \) links the utility function to the investor’s optimal payoff function. For any investor with an ART decreasing with time at the rate \( \alpha(t) \geq 0 \) the APPI strategy replicates the corresponding optimal payoff profile. The APPI payoff \( A_\alpha \) is therefore the optimal payoff for HARA investors with ART parameters

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\(^7\)Economists often use an additive lifetime utility that is the sum of the yearly utilities of consumption (see Gollier, 2001). Assuming log-normally distributed prices and constant consumption Merton (1971) also finds that for investors who have constant additive HARA type utilities the optimal payoff function is a power of the underlying plus a floor, i.e. a CPPI payoff.
\( \lambda(t) \) and \( \eta(t) \) decreasing during the averaging period at the rate \( \alpha(t) \).

![Figure 5: ART for \( \tilde{\alpha}(t) \) as a function of excess wealth for \( \lambda_0 = 0.2, \eta_0 = 2, T = 10 \) and \( \tau = 5 \)](image)

We illustrate the effect of changing \( \lambda(t) \) and \( \eta(t) \) on the ART at various times as a function of wealth for the standard geometric average in Figure 5. Maturity is \( T = 10 \) and the averaging period \( \tau = 5 \). The ART is linear in wealth, which is the case only for HARA utilities (see Gollier, 2001). The ART function has its root at \( F^* \). When the utility parameters change over time the slope of the ART function decreases, indicating increasing risk aversion, but the floor stays at its initial value.

Decreasing \( \eta(t) \) faster than \( \lambda(t) \) would increase risk tolerance for low levels of wealth and result in a lowering of the floor, which is always possible. On the other hand, decreasing \( \lambda(t) \) faster than \( \eta(t) \) would necessitate raising the floor of the optimal strategy faster than at the risk-free rate and could eventually force the entire portfolio into the risk-free asset.

6. Performance Comparison of APPI Strategies

To illustrate the effects of averaging and the differences between various APPI parameters, we simulate one path of a risky asset price following a GBM over 10 years. Throughout this section we take the standard geometric Brownian (4) as the underlying asset. We use the characteristics of a typical equity index with \( \mu = 5\% \) and \( \sigma = 18\% \).\(^8\)

\(^8\)These are approximately the mean and standard deviations of the S&P500 from 1990 to 2010.
Figure 6 shows the discounted risky asset price together with the discounted paths of APPI strategies for various parameter combinations. The risk-free rate is again \( r_f = 3.5\% \). The standard CPPI strategy has only two strategy parameters, the floor and the multiplier, but an APPI investor additionally needs to specify the start of the averaging period. We show three different choices of averaging periods: \( \tau = 8 \), 5 and 2 years. The initial fund value is 1 and maturity is 10 years. Figure 6(a) shows the strategy with the greatest volatility as the floor is low and the multiplier high. In the first 2 years the portfolio values move like the underlying until investors with the longest averaging period \( \tau = 8 \) start averaging, which reduces their portfolio volatility. None of the strategies dominates the others; the path of the underlying price determines which strategy has the highest final fund value. In this instance, as the underlying price decreases at the start of the averaging periods, the strategies with the longest averaging periods happen to outperform the strategies with the shorter averaging periods.
Comparing Figures 6(a)–6(d), we see that the volatility of the fund value is high when either
the multiplier is high, the floor is low, or both. The strategies in Figures 6(c) and 6(d),
which have low multipliers have the lowest volatilities and the fund values grow at about
the risk-free rate. Naturally, the APPI strategies with the highest volatility have also the
greatest upside potential.

Figure 7: Excess fund value densities of APPI strategies on a standard geometric average
for T=1 and several averaging periods for F=0.8 and m=2

Figure 7 shows the probability densities of the excess fund value above initial wealth of APPI
strategies for various averaging periods. The time horizon is 1 year and the initial fund value
is 1; we set the floor at 0.8 and the multiplier at 2. First we see that the Full APPI strategy
with \( \tau = 10 \) has the lowest volatility, skewness and kurtosis and therefore the highest peak.
At the other extreme, the standard CPPI strategy with \( \tau = 0 \) has the lowest peak. The
densities of the strategies in Figure 7 approach the Full APPI when the averaging period is
long and approach the standard CPPI when it is short.

7. Conclusion

We have defined an average portfolio insurance payoff that is the sum of a floor growing
at the risk-free rate plus a power of a generalised geometric average of the underlying risky
asset price. This payoff combines the advantages of average price options with portfolio
insurance strategies. The investor can choose the averaging period to be less than or equal
to the investment horizon. Whereas a CPPI strategy maintains the exposure to the risky asset at a constant multiple of the buffer, the multiplier in an APPI strategy decreases at the rate $\alpha(t)$ with time, reducing the share of the portfolio invested in the risky asset. For a standard geometric average the share decreases linearly to zero at maturity.

These characteristics make APPI strategies suitable for long-term investors who save for their pension. We use a general geometric average $G_\alpha$ which weighs the log-prices of the underlying asset $\ln(S(t))$ with $-\alpha(t)$ giving the standard geometric average when $-\alpha(t) = \frac{dt}{t}$. We prove that APPI strategies defined on a generalised average $G_\alpha$ are optimal implementations of life-cycle investments for investors with time-varying HARA utility functions such that both the local coefficient of absolute risk tolerance and the sensitivity of risk tolerance to wealth decrease with time at the same rate $\alpha(t)$. Together, the investor’s risk tolerance parameters and the market parameters determine the floor and multiplier specifying the optimal APPI payoff. Thus, APPI strategies can be manufactured to optimally match investors’ individual degrees of risk aversion.

APPI payoffs need to be defined for a certain maturity. Financial intermediaries could offer them to long-term investors in the same way as they offer other structured products. Alternatively, long-term investors may choose to replicate these payoffs by implementing a dynamic investment strategy. To help manufacture APPI payoffs and assess the risk of a portfolio containing such structured products, we examine the sensitivities of their fair prices to changes in the underlying asset price and its volatility. We compare the sensitivities of an APPI product to those of a similar fixed-maturity CPPI product with same maturity, floor, multiplier and initial value. The gamma and vega of the APPI product, before and during the averaging period, are lower than the gamma and vega of the similar CPPI product. Finally, we use Monte Carlo simulations to illustrate how the choice of floor and multiplier influences the performance of an APPI strategy.
References


