Numerical Methods with Lévy Processes

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Objective:

i) Find models of asset returns, etc
ii) Get numbers out of them.

Why?

VaR and risk management
Valuing and hedging derivatives
Why not?

Usual assumption:

Returns are normally distributed,
Joint distributions are normal.

Alas, not true.

Marginal distributions:	Not normal
Joint distributions:	Not jointly normal.
	(not a normal copula).

Alas, not ignorable.

Effects are too great.

Modelling joint returns distributions

M assets, S^i_t, $i = 1,\ldots,M$,
Returns, $R^i_t = \ln(S^i_t/S^i_0)$.

Marginal distributions:

$$F^i_t(r) = \Pr[R^i_t < r],$$

Joint distribution:

$$F_t(r^1,\ldots, r^M) = \Pr[R^1_t < r^1, \ldots , R^M_t < r^M],$$

Separate out the marginals and the dependency.

Note: $F^i_t : \mathbb{R} \to [0,1]$, $u^i_t = F^i_t(R^i_t)$ is uniform.
Use each F^i_t to map F_t onto
a joint distribution function C on $I^M = [0,1]^M$,

$$F_t(r^1,\ldots, r^M) = C_t(F^1_t(r^1),\ldots, F^M_t(r^M)).$$

$C : I^M \to [0,1]$ is a probability distribution on I^M with uniform marginals.

C is called a copula.

Jointly normal distributions?

$F^i_t(r)$ are normal,

C_t is the normal copula.

Generalise:

Model marginals as Lévy processes

Choose a non-normal copula.

This talk:

The problems and joys of Lévy processes.

Lévy Processes

$X = \{X_t\}_{t \geq 0}$, $X_0 = 0$, is a Lévy process if

i) 'Increments are independent of the past':

$\forall\ 0 \leq s < t < \infty$, $X_t - X_s$ is independent of $\mathfrak{I}_s$

ii) 'Increments are stationary':

$\forall\ 0 \leq s < t < \infty$, the distribution of $X_t - X_s$ is the same as the distribution of X_{t-s}.

iii) X_t is continuous in probability:

(ie $\forall\ \varepsilon > 0$, $\Pr[\ |X_t - X_s| > \varepsilon\ ] \to 0$ as $s \to t$)

X_t will have a modification which is càdlàg.
This is 'the' Lévy process.

Additive process: has properties (i) and (iii)

The Lévy-Khintchine representation

Characteristic function of μ, a prob measure on R^d:

$$\hat{\mu}(z) = \int_{R^d} e^{iz'x}\mu(dx) = E^{\mu}[e^{iz'x}],\ \ z \in R^d.$$

If X_t is a Lévy process and $X_1|X_0 \sim F_{X_1} = \mu$, then

$$\hat{\mu}(z) = \exp(\ \phi(z)\ ),\ \ z \in R,$$

with

$$\phi(z) = -\tfrac{1}{2}z'Az + iz'\gamma + \int_{R^d} (e^{iz'x} - 1 - iz'x.1_D(x))\nu(dx).$$

where $D = \{\ x\ |\ |x| \leq 1\}$ is the unit ball in R^d, and

- A is a symmetric non-negative definite matrix,
- $\gamma \in R^d$,
- ν is a measure on R^d, such that
  - $\nu\{0\} = 0$,
  - $\int_{R^d} (|x|^2 \wedge 1)\nu(dx)\ ) < \infty$,

(A,ν,γ) is the generating triplet of μ

ν isn't a probability measure. May not be integrable.

i) (A,ν,γ) is unique

ii) $(A,\nu,\gamma) \leftrightarrow$ Lévy processes

Notes:

If $\mu \leftrightarrow X_1$ has generating triplet (A,ν,γ) then
$\mu^t \leftrightarrow X_t$ has generating triplet $(tA,t\nu,t\gamma)$.

ν is called the Lévy measure of μ.
If $\nu(dx) = k(x)dx$ has a density,
k is called the Lévy density of μ.

The Lévy-Khintchine representation is not unique.
Can have:
$\phi(z) = -\frac{1}{2}z'Az + iz'\gamma_c + \int_{R^d} (e^{iz'x} - 1 - iz'x.c(x))\nu(dx)$.
where, eg,
$c(x) = (1 + |x|^2)^{-1}$,
$c(x) = 1_{\{|x|\leq\varepsilon\}}(x)$, $\varepsilon > 0$, etc
when
$\gamma_c = \gamma + \int_{R^d} x(c(x) - 1_D(x))\nu(dx)$.
Write $(A, \nu, \gamma_c)_c$.

Centre and Drift

Suppose that $\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty$,

then can set $c(x) = 0$ and

$\phi(z) = -\frac{1}{2} z' A z + i z' \gamma_0 + \int_{R^d} (e^{iz'x} - 1) \nu(dx)$.

This γ_0 is the drift of μ.

Suppose that $\int_{|x| > 1} |x|^2 \nu(dx) < \infty$,

then can set $c(x) = 1$ and

$\phi(z) = -\frac{1}{2} z' A z + i z' \gamma_1 + \int_{R^d} (e^{iz'x} - 1 - iz'x) \nu(dx)$.

This γ_1 is the centre of μ.

If γ_1 exists then $\gamma_1 = \int_{R^d} x \, \mu(dx)$ is the mean of μ.

μ Gaussian then $\nu = 0$ and $\gamma_0 = \gamma_1$.

Brownian motion with drift,

γ_0 is the drift of the Brownian motion.

Observations

μ compound Poisson,

jumps arriving at a rate c,

jump sizes distributed according to σ,

then $A = 0$, $\nu = c\sigma$, $\gamma_0 = 0$.

Γ-distribution, parameters c, $\alpha > 0$, then

$$\phi(z) = c\int_{[0,\infty)}(e^{ixz} - 1)\frac{e^{-\alpha x}}{x}dx,$$

so $A = 0$, $\nu(dx) = c\frac{e^{-\alpha x}}{x}1_{[0,\infty)}(x)dx$, $\gamma_0 = 0$.

This ν has infinite mass.

X_t additive, continuous sample paths as

iff X_t has Gaussian distribution $\forall t$,

ie, X_t is Brownian motion.

A is Gaussian covariance of μ.

$\nu = 0$ iff μ is Gaussian.

$A = 0$, then is purely non-Gaussian

$A, \gamma = 0$, then μ is pure jump.

Connections

X_t a Lévy process on R^d, generating triple (A,ν,γ).

Is type A: if $A = 0$, $\nu(R^d) < \infty$,

type B: if $A = 0$, $\nu(R^d) = \infty$, $\int_{|x| \leq 1}|x|\nu(dx) < \infty$,

type C: if $A \neq 0$, or $\int_{|x| \leq 1}|x|\nu(dx) = \infty$.

Sample paths of X_t are:

cts iff $\nu = 0$,

Piecewise constant iff

i) X_t is type A with $\gamma_0 = 0$, or

ii) X_t is compound Poisson

$\nu(R^d) = \infty$, then

jump times are countable, dense in $[0,\infty)$.

$0 < \nu(R^d) < \infty$, then

jump times are countable, but not dense as.

Time to first jump is exponential, mean $\nu(R^d)^{-1}$.

X_t is type A or B

then has finite variation on $(0,t]$, as.

X_t is type C

then has infinite variation on $(0,t]$, $\forall$ t.

Modelling with Lévy Processes

Stock Model, eg

Under risk-neutrality suppose that

$S_t = S_0 \exp(rt + \sigma X_t - \omega t)$,

where r is the short rate and

ω compensates for the drift in X_t,

so that $S_t/\exp(rt)$ is a martingale,

$E[e^{\sigma X_t}] = \exp(t\omega)$

(ie, X_t specified under the pricing measure).

Interest Rate Model, eg

$dr_t = \alpha(\mu(t) - r_t)dt + \sigma dX_t$,

Note: In each case can assume that X_1 has zero mean and unit variance in unit time.

Need to price options, etc, on S_t, r_t, etc

How to Solve?

Gaussian case:

- Explicit/analytic solutions?
- PDE
- Monte Carlo integration
- Lattices

Lévy case:

- Analytic solutions:
  - May involve tricky numerical integration
- PDE:
  - Method of lines? Is it general?
- Monte Carlo:
  - Tricky if Lévy density unbounded near zero.
- Lattices:
  - Need very high order branching?

Problem: Too many small jumps
Too many big jumps

The Generalised Hyperbolic Distribution

(Barndorff-Nielsen (01), Eberlein (01), Rydberg (99))

The density is:

$f_{GH}(x|\lambda,\alpha,\beta,\delta,\mu)$

$$= \frac{\left(\alpha^2-\beta^2\right)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}.(\delta^2 + (x-\mu)^2)^{(\lambda-½)/2}$$

$$\times K_{\lambda-½}(\alpha(\delta^2+(x-\mu)^2)^{-½})\exp(\beta(x-\mu)),$$

where $K_\nu(z) = ½\int_0^\infty y^{\nu-1}\exp\left(-\frac{1}{2}z\left(y+y^{-1}\right)\right)dy$

is the modified Bessel function of the third kind.

Parameters:	$\alpha > 0$,	shape,
	$0 \le \beta < \alpha$,	skewness,
	$\lambda \in R$,	class of the distribution,
	$\mu \in R$,	location,
	$\delta > 0$,	scale.

Reparameterise: replace α, β by

$\xi = (1 + \delta\sqrt{\alpha^2-\beta^2})^{-½}$, $\chi = \xi\beta/\alpha$, so $0 \le |\chi| < \xi < 1$.

ξ and χ are invariant under $X \to aX + b$.

l = 1: Hyperbolic distribution

Then $K_{½}(z) = (\pi/2z)^{½}e^{-z}$, and

$f_H(x) = f_H(x|\alpha,\beta,\delta,\mu) = f_{GH}(x|1,\alpha,\beta,\delta,\mu)$

$$= \frac{\left(\alpha^2-\beta^2\right)^{1/2}}{\alpha\delta K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\exp(-\alpha(\delta^2+(x-\mu)^2)^{½} + \beta(x-\mu))$$

Centred if $\mu = 0$, symmetric if $\beta = 0$.

Get special cases (ξ - χ parameterisation):

$\xi \to 0$,	Normal
$\xi \to 1$,	Laplace
$\chi \to \pm\xi$,	Generalised inverse Gaussian
$\lvert\chi\rvert \to 1$,	Exponential

l = -½: Normal Inverse Gaussian distribution

$f_{NIG}(x|\alpha,\beta,\delta,\mu) = f_{GH}(x|-½,\alpha,\beta,\delta,\mu)$

$$= \frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}$$

Distribution of first hitting times of a 2-dim BM, starting at $(\mu,0)$ to $R\times\{\delta\}$, drift $(\beta,\sqrt{\alpha^2-\beta^2})$, vol (1,1).

Variance-gamma

Special case of the GH distribution

$$f_{VG}(x \mid \sigma, \nu, \mu) = f_{GH}\left(x \mid \frac{\sigma^2}{\nu}, \sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^4}}, \frac{\theta}{\sigma^2}, 0, 0\right)$$

Explicitly this becomes:

$$f_{VG}(x|t,\mu,\nu) = \Gamma(\mu^2 t/\nu, \mu/\nu)$$

$$= \left(\frac{\mu}{\nu}\right)^{\mu^2 t/\nu} \frac{x^{\left(\mu^2 t/\nu\right)-1}}{\Gamma\left(\mu^2 t/\nu\right)} \exp\left(-\frac{\mu}{\nu} x\right), \quad x > 0$$

Convolutions:

NIG and VG processes: closed under convolutions

$$f_{NIG}(\alpha,\beta,\delta_1,\mu_1) * f_{NIG}(\alpha,\beta,\delta_2,\mu_2) = f_{NIG}(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)$$

Lévy Processes and Time Changes

X_t a 1-dimensional semi-martingale:
Representable as a time-changed Brownian motion.

$X_t = w_{h(t)}$, w_t a Brownian motion
h(t) a (stochastic) time change.

Lévy Processes are semimartingales…

Variance gamma process:
Brownian motion subordinated to Γ.

$x_t \sim \Gamma(t,\nu t)$ a gamma variate, density

$$f(x_t) = \frac{x^{(t/\nu)-1} e^{-x/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)},$$

Then if X_t is VG have:

$$X_t \equiv X_{VG}(t|\sigma,\nu,\theta) = \theta x_t + \sigma w_{x(t)}.$$

Normal inverse Gaussian process:
Brownian motion subordinated to IG.

$x_t \sim IG(\delta, \gamma)$ an inverse Gaussian variate, density

$$f_{IG}(x \mid \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{\gamma^2}{2x}\left(x - \frac{\delta}{\gamma}\right)^2 \right),$$

Then $X_t \equiv X_{IG}(t|\delta, \gamma) = w_{x(t)}$.

Subordinator Approach to Lévy Processes

Derivative, payoff H_T at time T, value c_t at time t.
Martingale valuation:

$$c_t = E_t[\ c_T\ ],$$

where

$$c_T = H_T p_t / p_T$$

and p_t is a numeraire,
E_t is expectations operator wrt p_t.

Suppose H_T and p_t depend on state variable S_t,
S_t depends upon a Lévy process X_t.

Basis of research approach:
Use subordinator representation of X_t, $X_t = w(h_t)$.
Iterated expectation:

$$c_t = E_t[\ c_T\ ] = E_t[\ E_t[\ c_T \mid h_t\ ]\ ].$$

Use numerical methods?
<u>Can value inner and outer expectations separately</u>.

Various approaches

Each expectation:

MC, PDE, lattice?

Appropriate, or not, for

P or NP: Path dependent options

A or NA: American or Bermudan options

C or NC: Ease of calibration

Overview of methods		**Outer**		
		MC	Lattice	PDE
Inner	MC	P, NA, NA	NP, A, NC	?
	Lattice	NP, ~A, C	NP, A, C	?
	PDE	?	?	?

MC + MC: Ribeiro and Webber

MC + lattice: Kuan and Webber (in progress)

MC + MC

Generate a path $H = \{h_j\}$for h_t by Monte Carlo.
Conditional on H, generate a path for X_t.

Straightforward. Trick is to apply speed-ups.

Ribeiro and Webber: Stratified sampling + bridge
VG process: working paper available
NIG process: in progress.
Get good speed-ups (up to a factor of ~800)

MC + Lattice

Kuan and Webber (in progress)
Generate a path $H = \{h_j\}$for h_t by Monte Carlo.
Conditional on H, generate a lattice for X_t.
Stock model: Easy to value
European options,
Barrier options
Interest rate model:
Can calibrate to a market term structure.

Simulating a Lévy Process

Discretise time: $0 = t_0 < \ldots < t_N = T$.
Set $\Delta t_j = t_{j+1} - t_j$.

Stratified sampling:
Suppose $X \sim F_X$.

$u \sim U[0,1]$ uniform then $(F_X)^{-1}(u) \sim F_X$.
u_i, $i = 1,\ldots,M$ a stratified sample from $U[0,1]$ then $(F_X)^{-1}(u_i)$ is a stratified sample from F_X.

Stratified sample of $U[0,1]$?
Let $v_i \sim U[0,1]$, $i = 1,\ldots,M$, then
$u_i = (i + v_i - 1)/M$ is a stratified sample of $U[0,1]$.

Lévy process X_t. Write $X_j \sim F_{t_j}$.

Suppose have a stratified sample $X_{i,N}$, $i = 1,\ldots,M$.
How to construct a set of paths $0 = X_{i,0},\ldots,X_{i,N}$ with correct conditional properties?

The bridge

Suppose $X \sim F_X$, $Y \sim F_Y$, $Z \sim F_Z$,
densities f_X, f_Y, f_Z,
and $Z = Y + X$

Suppose have a draw z of Z, what is distribution X|Z?

Write $f_{(X,Z)}(x,z)$ for the joint density of X and Z.

Have $f_{X|Z}(x) = f_{(X,Z)}(x,z)/f_Z(z)$

$= f_X(x).f_Y(z-x)/f_Z(z)$

if X and Y = Z - X are independent.

Our situation: $X = F_{t_i}$, $Y = F_{t_j}$, $Z = F_{t_i + t_j}$,
increments in a Lévy process X_t.

Application to a Gamma Process

Want a regular sample of a gamma process

$0 = h_0, \ldots, h_N$.

Are given $h_0 = 0$ and $h_N \sim \Gamma(t_N, \nu t_N)$, want h_i at time t_i.

Have $z \equiv h_N$, $x \equiv h_i$, $y \equiv h_N - h_i$.

Have

$$f_X(x) = \frac{x^{t_i/\nu - 1}}{\nu^{t_i/\nu}\Gamma(t_i/\nu)} \exp(-x/\nu)$$

$$f_Y(z\text{-}x) = \frac{(z-x)^{(t_N - t_i)/\nu - 1}}{\nu^{(t_N - t_i)/\nu}\Gamma((t_N - t_i)/\nu)} \exp\left(-\frac{z-x}{\nu}\right)$$

$$f_Z(z) = \frac{z^{t_N/\nu - 1}}{\nu^{t_N/\nu}\Gamma(t_N/\nu)} \exp(-z/\nu)$$

so

$$f_{X|Z}(x) = \frac{1}{z} \frac{\Gamma\left(\frac{t_N - t_i}{\nu} + \frac{t_i}{\nu}\right)}{\Gamma\left(\frac{t_i}{\nu}\right)\Gamma\left(\frac{t_N - t_i}{\nu}\right)} \left(\frac{x}{z}\right)^{\frac{t_i}{\nu}} \left(1 - \frac{x}{z}\right)^{\frac{t_N - t_i}{\nu} - 1}$$

The gamma bridge distribution

Change variable to $p = x/z$, then

$p \sim B(t_i/\nu, (t_N-t_i)/\nu)$

is a beta variate with parameters t_i/ν and $(t_N-t_i)/\nu$.

ie, given h_N and h_0, $p \equiv \dfrac{h_i - h_0}{h_N - h_0} \sim B(t_i/\nu, (t_N-t_i)/\nu)$

Procedure:

i) Generate $b_i \sim B(t_i/\nu, (t_N-t_i)/\nu)$
ii) Set $h_i = h_0 + b_i.(h_N - h_0)$
iii) Fill in the rest of the h_j by binary chop.

Stratify:

i) Get h_N by stratified inverse transform.
ii) Get b_i by stratified inverse transform

Note: Can't do a fully stratified sample.
Instead use low discrepancy sampling.

Results (for details see paper)

Note: True se ≠ MC internal se.
Estimate true se by replicating 100 times and finding actual sd of result.

Comparison of two MC methods.

Suppose: MC method 1 gives se σ_1 in time t_1,
MC method 2 gives se σ_2 in time t_2.

If: t is proportional to M
σ is O(-1/2) in M,

then

$$E_{1,2} = (\sigma_2^2.t_2)/(\sigma_1^2.t_1)$$

is the efficiency gain of method 1 over method 2.

Conclusions

To use Lévy processes need good numerics.

This area is very new with few results.

Have outlined a general approach.

Have results for part of this,
working on other aspects.

Much more to do.