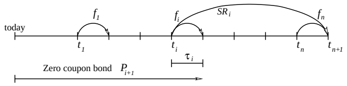


Peter Jäckel

LINKING CAPLET AND SWAPTION VOLATILITIES

Abstract

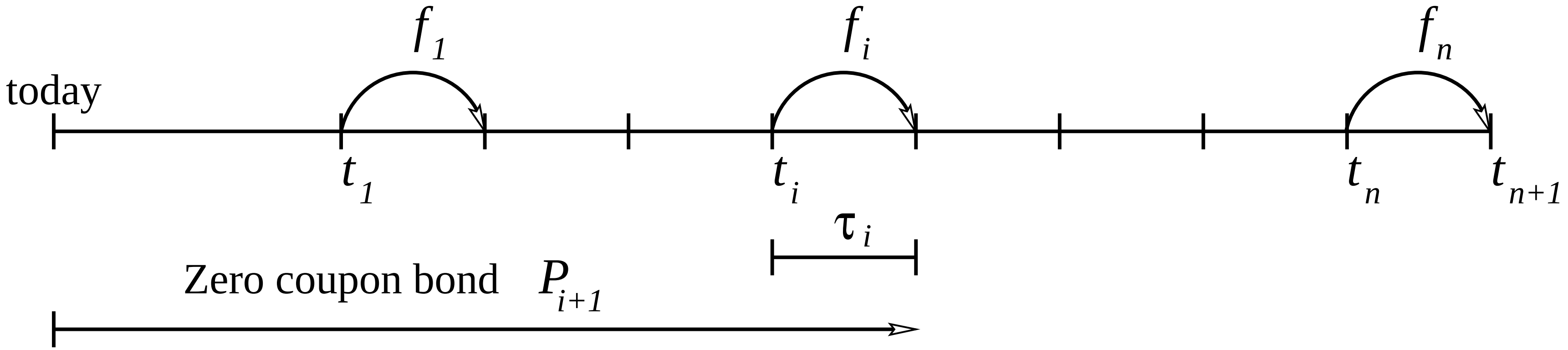
An approximation for the volatility of European swaptions is presented that makes it possible to calculate prices of swaptions without the need for numerical computations such as Monte Carlo simulations or lattice-based integration methods. The approximation can be used whenever the covariance matrix of an initial set of fixed income observables is known and thus applies to most interest rate models such as the extended Vasicek (also known as multi-factor Hull-White) model or the Brace-Gatarek-Musiela/Jamshidian framework. Also, the mechanism behind the remarkable accuracy of the approximation is explained.

Overview and Introduction

- Introduction and motivation
- Three different approaches to solving the problem
- The constant-weights (lowest order) approximation
- Justification of the integral approximation
- The shape correction as a result of the full partial differentiation
- Specific functional forms
- Results

Introduction and motivation

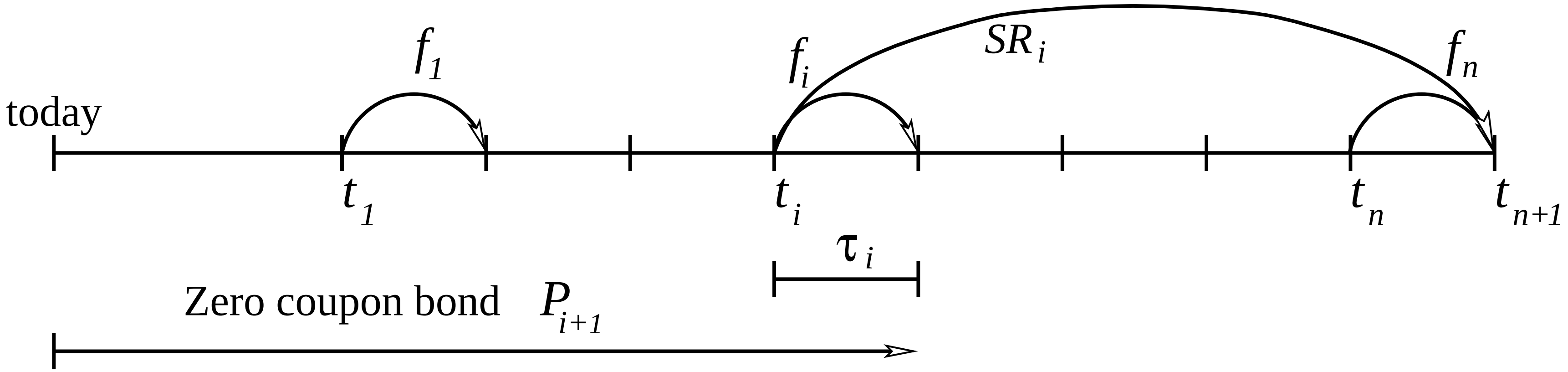
Given the dynamics of a set of spanning forward rates,



which is to mean that we have knowledge of the covariance matrix of the forward rates f_i or their logarithms $\ln f_i$,

$$\frac{\mathrm{d}C_{ij}^{\mathsf{FRA}}}{\mathrm{d}t}\,\mathrm{d}t \quad = \quad \langle \mathrm{d}\ln f_i \cdot \mathrm{d}\ln f_j \rangle \quad = \quad \sigma_i \sigma_j \rho_{ij}\,\mathrm{d}t \qquad (1)$$

can we infer (an approximation for) the dynamics of (forward) swap rates?

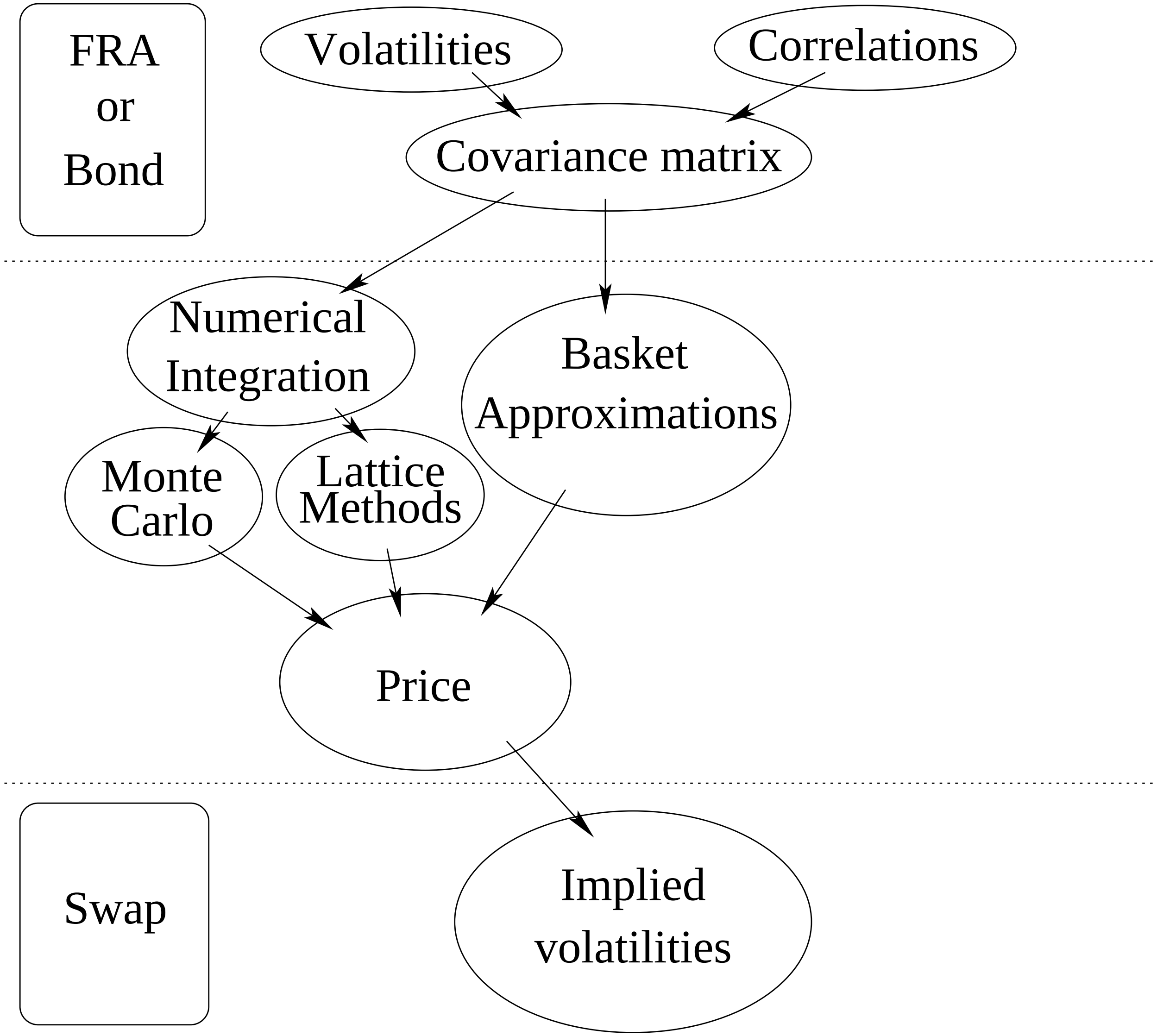


Of particular interest is the joint terminal distribution of a whole set of swap rates and/or forward rates because this allows the calculation of the values of market instruments and their correlations.

Examples:

- implied volatilities of a set of coterminal swaptions for calibration when pricing Bermudan swaptions or other callable structures
- FRA/swap correlations and their prices for calibration when valuing trigger swaps
- implied volatilities of a set of constant maturity swap rates for calibration when pricing CMS swaps or caps

Three different approaches to solving the problem



FRA
or
Bond

Volatilities

Correlations

Covariance matrix

Numerical
Integration

Basket
Approximations

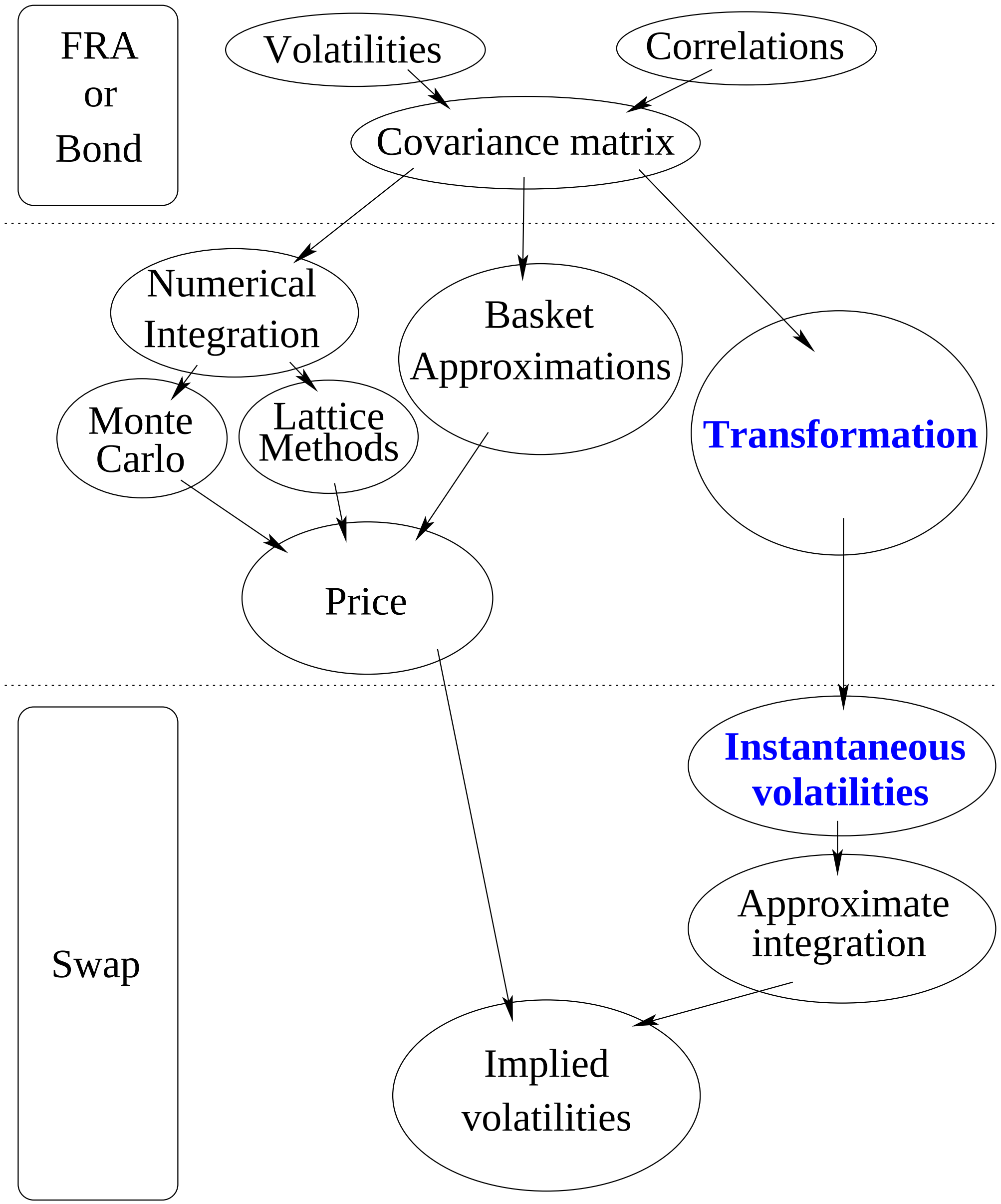
Monte
Carlo

Lattice
Methods

Price

Swap

Implied
volatilities



FRA
or
Bond

Volatilities

Correlations

Covariance matrix

Numerical Integration

Basket Approximations

Transformation

Monte Carlo

Lattice Methods

Price

Instantaneous volatilities

Swap

Approximate integration

Implied volatilities

Three different approaches to solving the problem

The standard methods for numerical integration are well known:

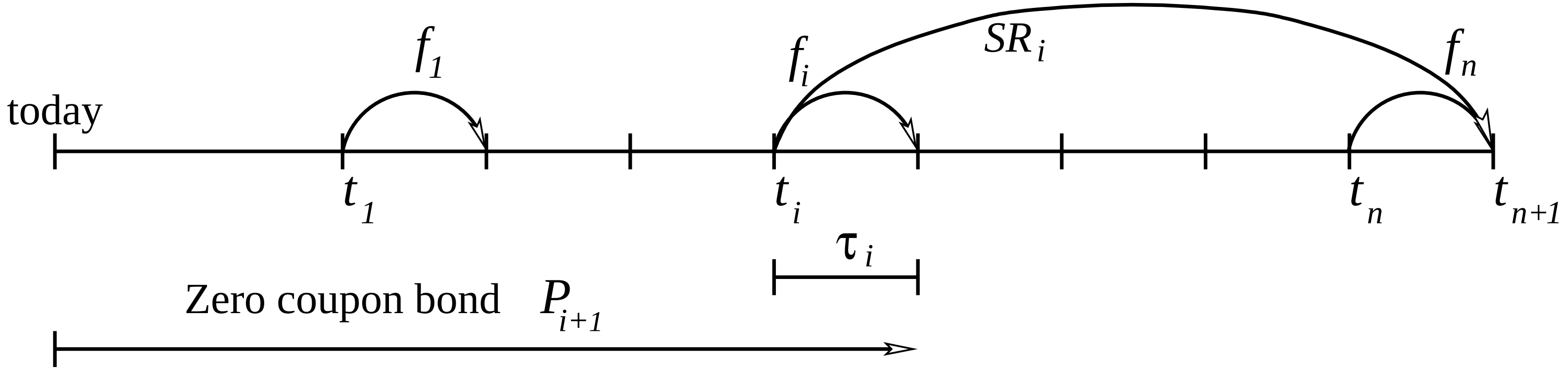
- Monte Carlo simulation,
- Multi-factor trees,
- adaptive lattice integration,
- finite difference, and
- finite element methods.

As for basket-style approximations, there are

- the Turnbull-Wakeman [TW91] method (matching of two moments with a lognormal density),
- the matching of three moments with a displaced lognormal density (also known as Johnson distribution),
- approximate integration conditioned on the geometric mean [Cur99],
- expansion techniques [Ju01, RDK01],
- and many other algorithms [Jam89].

The constant-weights (lowest order) approximation

Let $SR_{i\times(n-i)}$ denote the par coupon rate of a swap starting at time t_i with terminal maturity t_{n+1}, or SR_i for short in the following.



Then,

$$SR_i(t) = \sum_{j=i}^{n} w_{ij} f_j(t) \qquad \text{with} \qquad w_{ij} = \frac{P_{j+1}\tau_j}{\sum\limits_{k=i}^{n} P_{k+1}\tau_k} \ . \tag{2}$$

The lowest order or *constant weights* approximation for the volatility σ_i^{Swap} of SR_i is given by

$$\left[\sigma_i^{\text{Swap}}\right]^2 = \frac{\sum\limits_{j,k=i}^{n} w_{ij} w_{ik} f_j f_k \frac{\mathrm{d}C_{jk}^{\text{FRA}}}{\mathrm{d}t}}{\left[\sum\limits_{j=i}^{n} w_{ij} f_j\right]^2} \quad . \tag{3}$$

Remember, this is still only an approximation for the *instantaneous* swap rate volatility, which is subject to stochastic variability due to the occurences of the forward rates in (3), i.e.

$$w_{ij} = w_{ij}(t) \quad \text{and} \quad f_i = f_i(t)\ , \quad \text{etc} \quad .$$

In order to obtain an approximation for the *implied* volatility, set

$$\left[\hat{\sigma}_i^{\text{Swap}}\right]^2 \cdot t_i \quad = \quad \frac{\sum\limits_{j,k=i}^{n} w_{ij}(0)\, w_{ik}(0)\, f_j(0)\, f_k(0)\, C_{jk}^{\text{FRA}}(t_i)}{\left[\sum\limits_{j=i}^{n} w_{ij}(0)\, f_j(0)\right]^2} \tag{4}$$

with

$$C_{jk}^{\text{FRA}}(t_i) \;=\; \int\limits_{t=0}^{t_i} \sigma_j^{\text{FRA}}(t)\sigma_k^{\text{FRA}}(t)\rho_{jk}^{\text{FRA}}\,\mathrm{d}t \quad . \tag{5}$$

Justification of the integral approximation

How can we justify to use the initial yield curve for the calculation of the coefficients

$$\zeta_{ijk}(t) \quad = \quad \frac{w_{ij}(t)\, w_{ik}(t)\, f_j(t)\, f_k(t)}{\left[\sum_{j=i}^{n} w_{ij}(t)\, f_j(t)\right]^2} \tag{6}$$

that determine the instantaneous volatility

$$\left[\sigma_i^{\text{Swap}}(t)\right]^2 \quad = \quad \sum_{j,k=i}^{n} \zeta_{ijk}(t) \frac{\mathrm{d}C_{jk}^{\text{FRA}}(t)}{\mathrm{d}t} \tag{7}$$

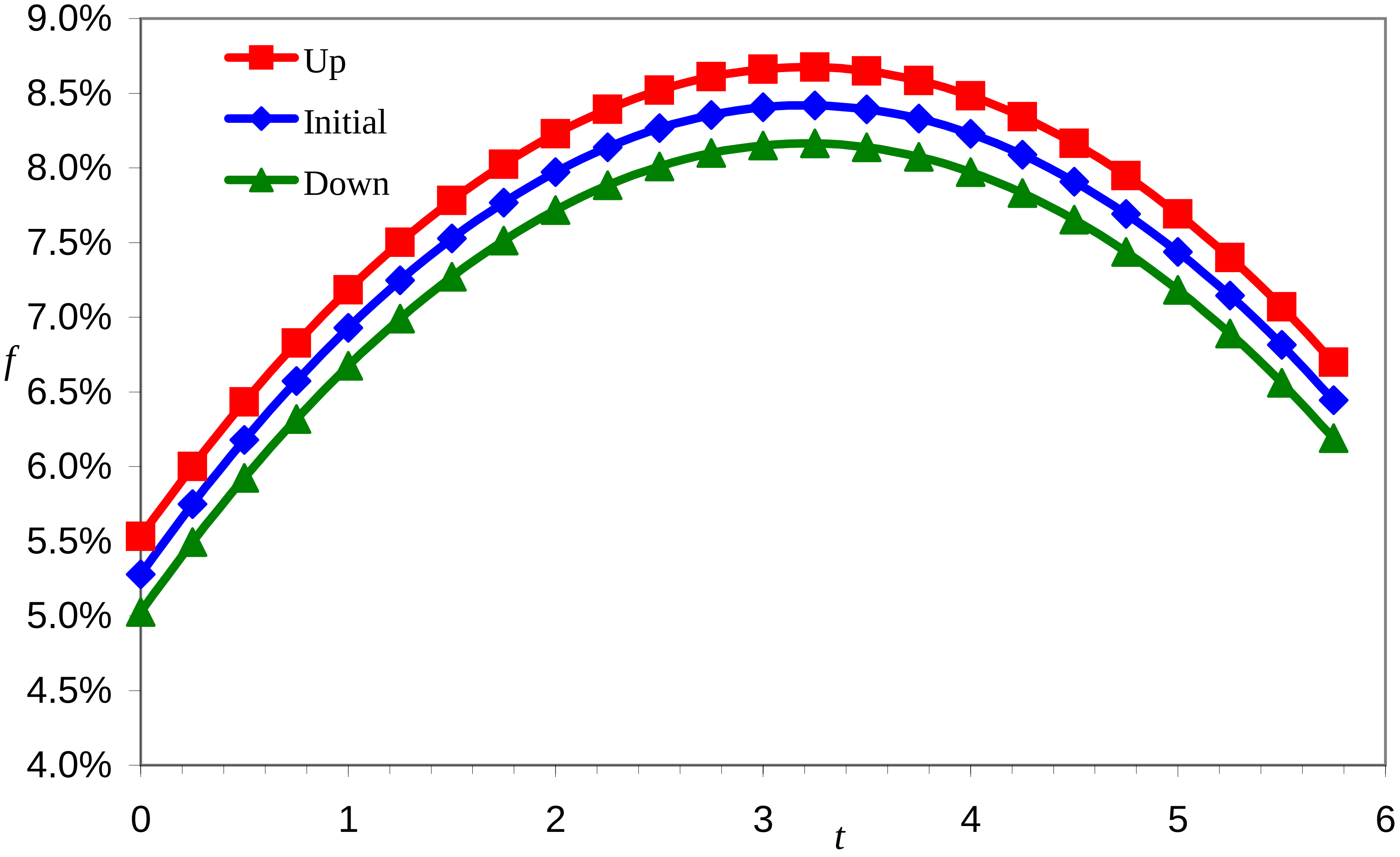
in the integral approximation (4) ?

The reason lies with the following two facts:

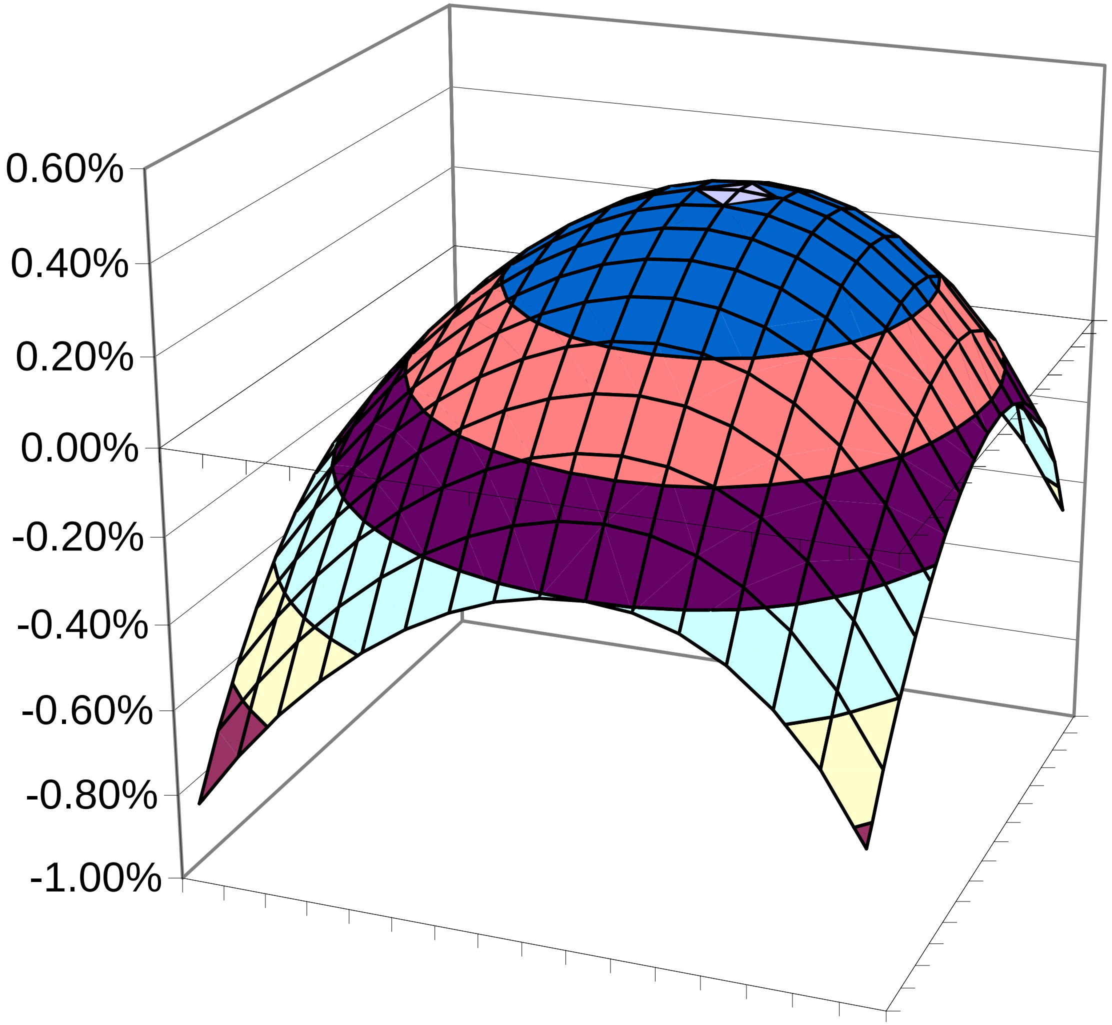
- the transformation coefficients ζ_{ijk} depend only very weakly on (log-)parallel perturbations of the yield curve given by the set of forward rates $\{f_i\}$. Since (log-)parallel shifts of the yield curve tend to have the highest partial variance associated with them in market-calibrated models, the most likely changes of the yield curve only result in very small changes of the transformation coefficients.

- equiprobable changes in the yield curve such as up-tilts and down-tilts partially cancel out the effective changes in the average experienced volatility of the swap rate.

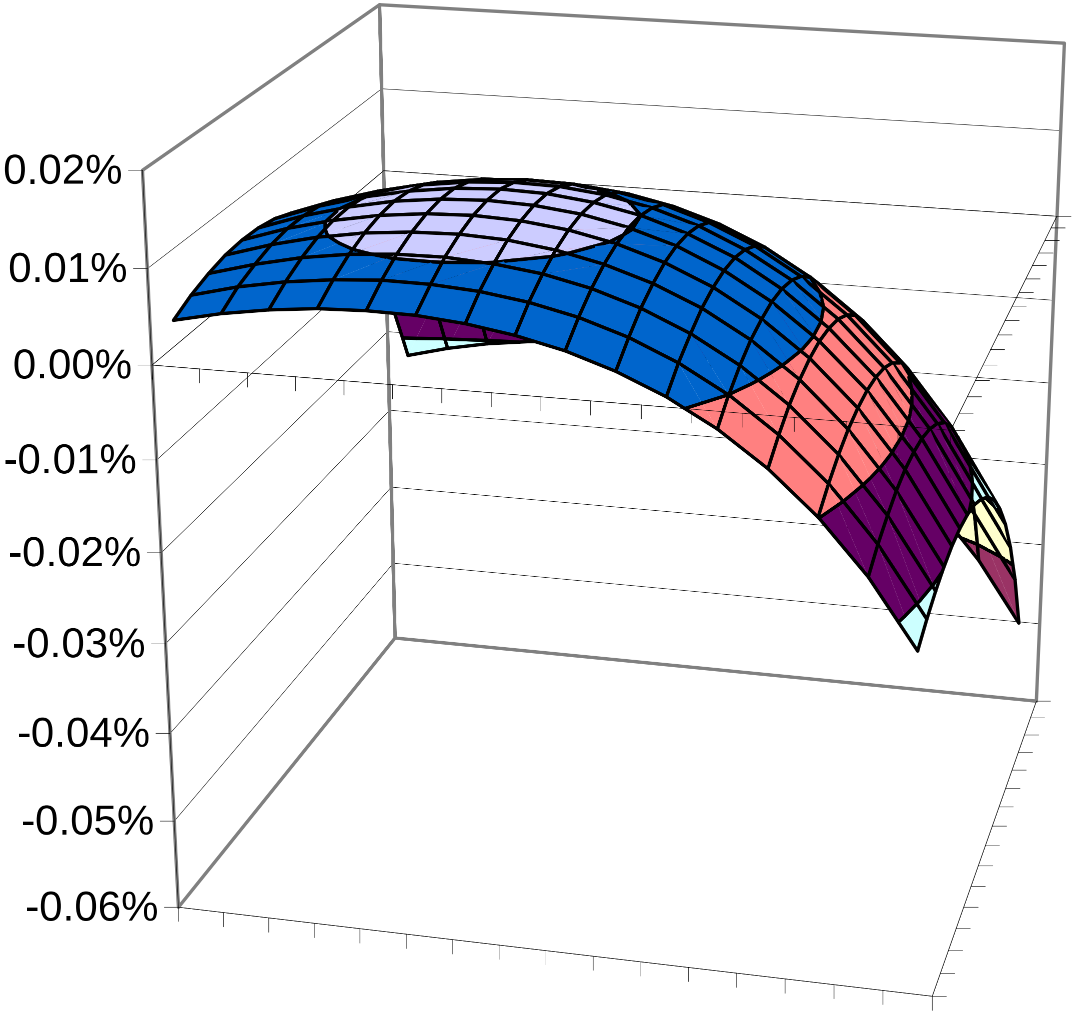
Example for shifts of the yield curve:



The resulting (relative) changes in the coefficients ζ_{ijk} for $i = 0$ from the up-shift:



The average (relative) changes in the coefficients ζ_{ijk} caused by the equiprobable up- and down-shift:



The shape correction as a result of the full partial differentiation

Define the coterminal floating leg values A_i as

$$A_i = \sum_{j=i}^{n} P_{j+1} f_j \tau_j N_j \quad \text{for} \quad i = 1 \ldots n \tag{8}$$

where N_j is the notional associated with accrual period τ_j. Also, define the set of coterminal annuities B_i by

$$B_i = \sum_{j=i}^{n} P_{j+1} \tau_j N_j \quad \text{for} \quad i = 1 \ldots n \tag{9}$$

which immediately leads to the swap rates SR_i

$$SR_i = \frac{A_i}{B_i} \;. \tag{10}$$

The elements of the swap rate covariance matrix C^{SR} can be written as

$$
\begin{aligned}
C^{SR}_{ij} &= \left\langle \frac{\mathrm{d}SR_i}{SR_i} \cdot \frac{\mathrm{d}SR_j}{SR_j} \right\rangle \\[2ex]
&= \sum_{\substack{k=1\\l=1}}^{\substack{n\\n}} \frac{\frac{\partial SR_i}{\partial f_k} \cdot \frac{\partial SR_j}{\partial f_l}}{SR_i \cdot SR_j} \cdot f_k f_l \cdot \left\langle \frac{\mathrm{d}f_k}{f_k} \frac{\mathrm{d}f_l}{f_l} \right\rangle \\[2ex]
&= \sum_{\substack{k=1\\l=1}}^{\substack{n\\n}} \frac{\partial SR_i}{\partial f_k} \frac{f_k}{SR_i} \cdot C^{FRA}_{kl} \cdot \frac{f_l}{SR_j} \frac{\partial SR_j}{\partial f_l} \ . \qquad (11)
\end{aligned}
$$

Defining the elements of the matrix $Z^{FRA\to SR}$ by

$$Z_{ik}^{FRA\to SR} = \frac{\partial SR_i}{\partial f_k}\frac{f_k}{SR_i} \ , \tag{12}$$

the mapping from the FRA covariance matrix C^{FRA} to the swap rate covariance matrix C^{SR} can be seen as a matrix multiplication:

$$C^{SR} = Z^{FRA\to SR} \cdot C^{FRA} \cdot {Z^{FRA\to SR}}^{\mathsf{T}} \quad . \tag{13}$$

Once the transformation matrix Z is known, all other required calculations are reduced to linear algebra!

This key result holds for the transformation from FRA to swap rate covariances in the BGM/J (lognormal forward rates) model, the transformation from bond covariances to swap rate covariances or swap/FRA covariances in an extended Vasicek (generalised Hull-White) model, and many more!

As for the case of log-normal forward rates, using

$$\frac{\partial P_{i+1}}{\partial f_k} = -P_{i+1}\frac{\tau_k}{1+f_k\tau_k}\cdot \mathbf{1}_{(k\geq i)} \;, \tag{14}$$

where $\mathbf{1}_{(k\geq i)}$ is one if $k \geq i$ and zero otherwise, and using equations (8), (9), and (10), we have

$$\frac{\partial SR_i}{\partial f_k} = \left\{\frac{P_{k+1}\tau_k N_k}{B_i} - \frac{\tau_k}{1+f_k\tau_k}\cdot\frac{A_k}{B_i} + \frac{\tau_k}{1+f_k\tau_k}\cdot\frac{A_i B_k}{B_i^2}\right\}\cdot \mathbf{1}_{(k\geq i)} \;. \tag{15}$$

And finally,

$$Z_{ik}^{FRA\to SR} = \left[\underbrace{\frac{P_{k+1}N_k f_k\tau_k}{A_i}}_{\text{as in equation (3)}} + \underbrace{\frac{(A_iB_k - A_kB_i)f_k\tau_k}{A_iB_i(1+f_k\tau_k)}}_{\text{shape correction}}\right]\cdot \mathbf{1}_{(k\geq i)} \;. \tag{16}$$

I call the second term inside the square brackets of equation (16) the *shape correction*. Rewriting it as

$$\frac{(A_iB_k - A_kB_i)f_k\tau_k}{A_iB_i(1+f_k\tau_k)} = \frac{f_k\tau_k}{A_iB_i(1+f_k\tau_k)} \cdot \sum_{l=i}^{k-1}\sum_{m=k}^{n} P_{l+1}P_{m+1}N_lN_m\tau_l\tau_m(f_l - f_m) \tag{17}$$

highlights that it is a weighted average over inhomogeneities of the yield curve.

In fact, for a flat yield curve, all of the terms $(f_l - f_m)$ are identically zero and the mapping matrix $Z^{FRA\to SR}$ is equivalent to the constant-weights approximation (3).

In an extended Vasicek model, the forward bond ratios

$$F_i = \frac{P_i}{P_{n+1}} \tag{18}$$

evolve log-normally. The swap rate SR_i can be written as

$$SR_i = \frac{\sum\limits_{k=i}^{n} N_k (F_k - F_{k+1})}{\sum\limits_{k=i}^{n} N_k F_{k+1} \tau_k} \tag{19}$$

The same analysis as before gives

$$Z_{ik}^{F \to SR} = \frac{F_k}{\sum\limits_{m=i}^{n} N_m (F_m - F_{m+1})} \cdot \begin{cases} 0 & \text{for} \quad k < i \\ N_k & \text{for} \quad k = i \\ [N_k - N_{k-1}(1 + \tau_{k-1} SR_i)] & \text{for} \quad k > i \end{cases} \tag{20}$$

The approximate log-normal swap rate covariance matrix is thus

$$C^{SR} = Z^{F\to SR} \cdot C^{F} \cdot {Z^{F\to SR}}^{\mathsf{T}} \quad . \tag{21}$$

As before, approximate implied volatilities are simply given by using the integrated covariance matrix C^{F}.

However, since forward and swap rates are better represented by a normal distribution in the extended Vasicek model, it is more accurate to use

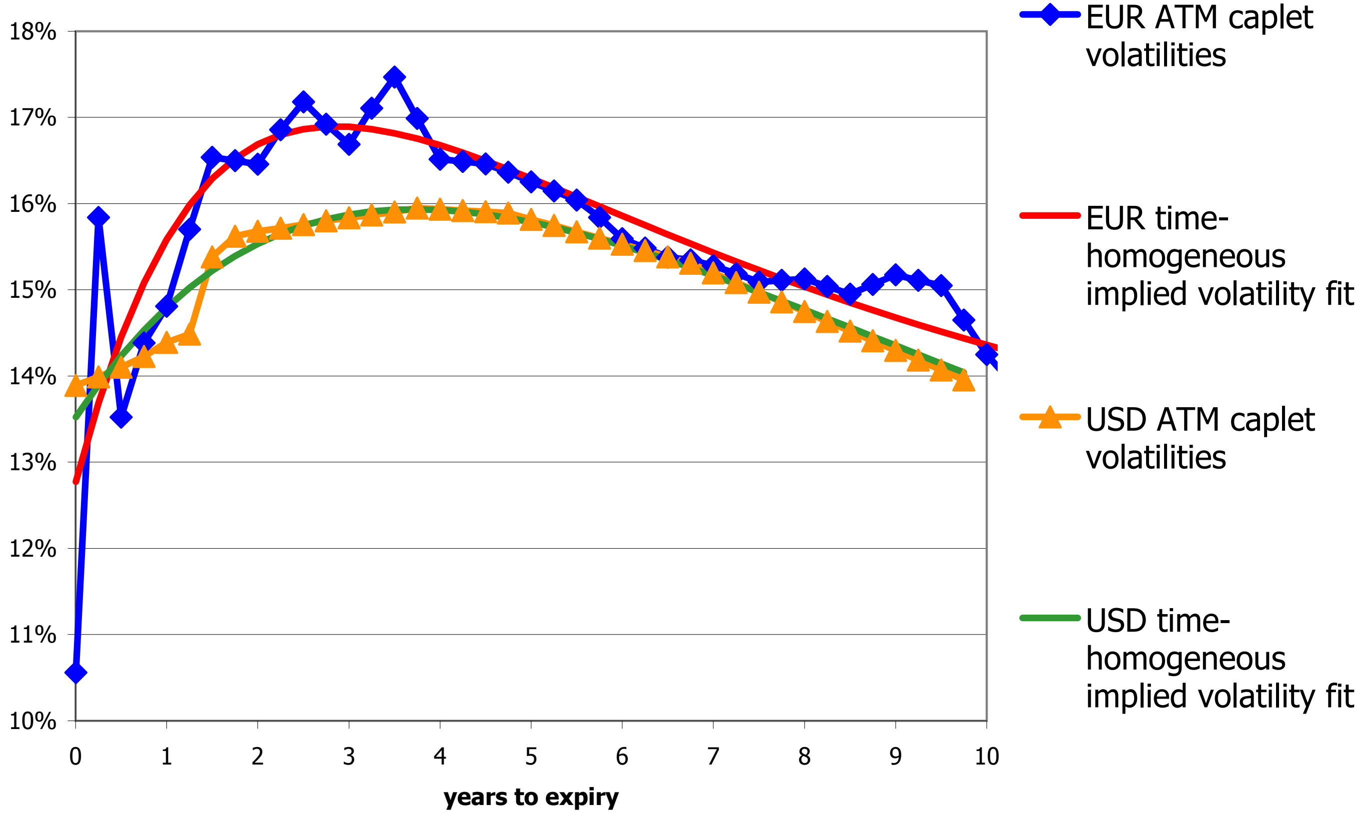
$$\hat{\sigma}_i^{SR} \cdot SR_i(0) \tag{22}$$

in a normal (Bachelier) rather than $\hat{\sigma}_i^{SR}$ in a log-normal (Black) option formula.

Specific functional forms

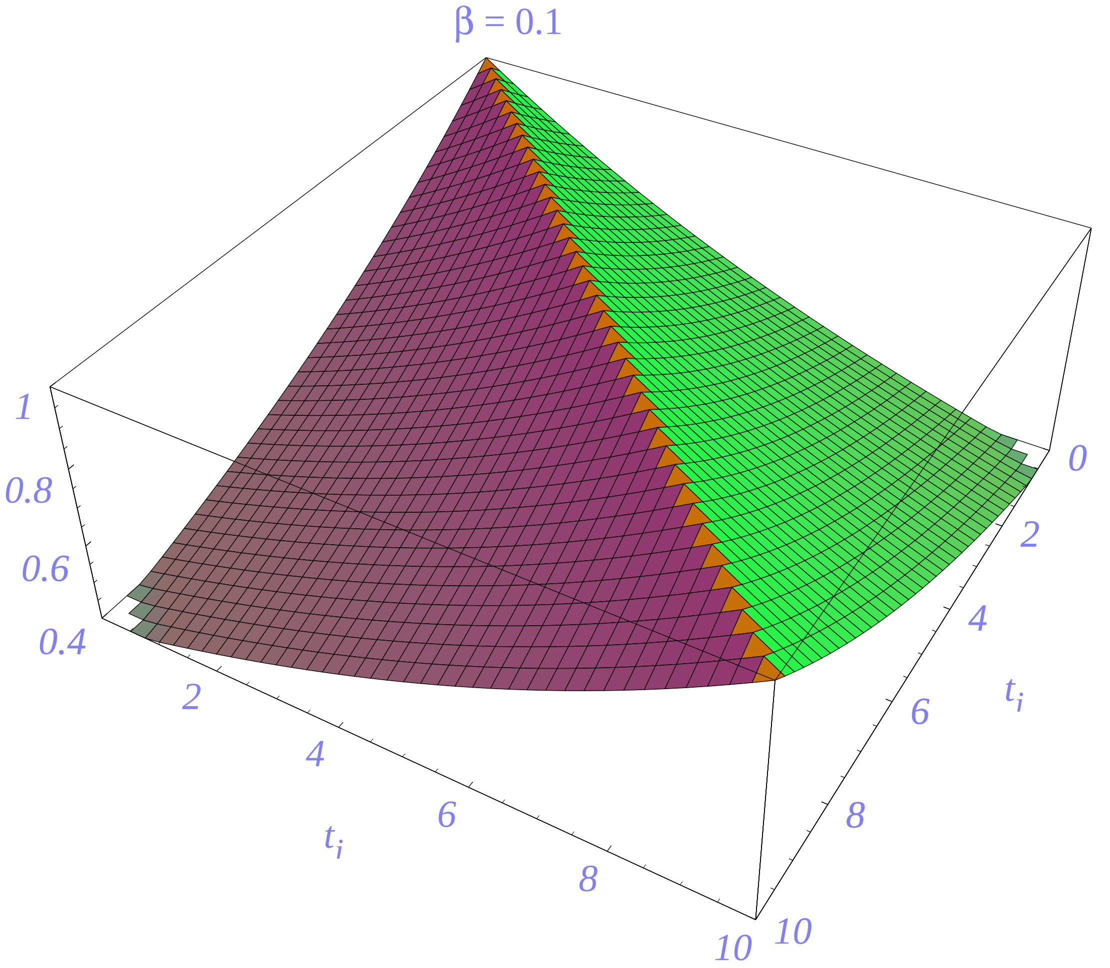
Using the time-homogeneous instantaneous forward rate volatility form

$$\sigma_j(t) = k_j \left[(a + b(t_j - t))\,\mathrm{e}^{-c(t_j - t)} + d \right] \tag{23}$$



and the instantaneous correlation form

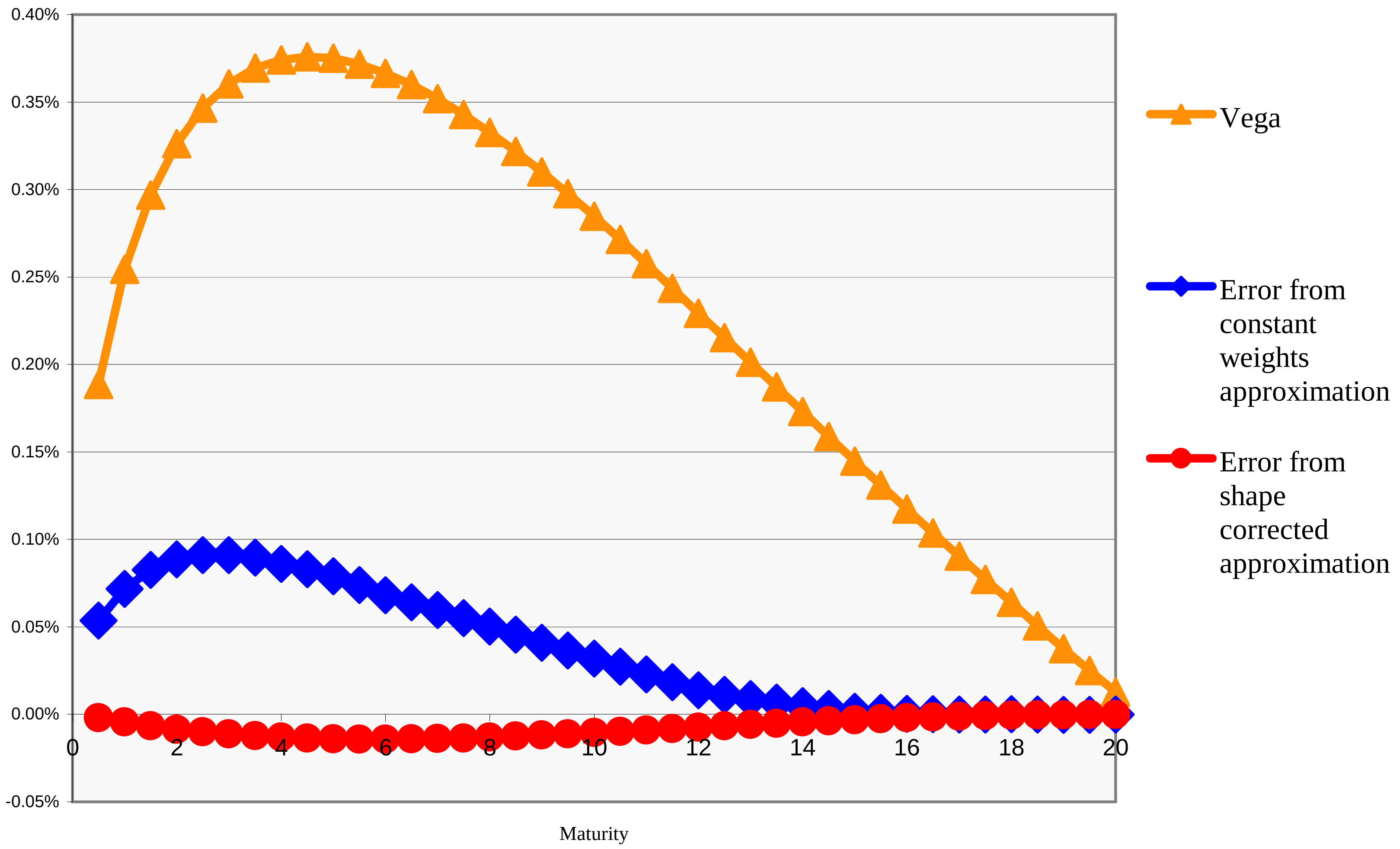
$$\rho_{jk} = \mathrm{e}^{-\beta\left|t_j - t_k\right|} \quad , \tag{24}$$



the indefinite integral of the covariance becomes

$$
\begin{aligned}
\int \rho_{ij}(t)\sigma_i(t)\sigma_j(t)dt \quad = \quad & \mathrm{e}^{-\beta\left|t_i-t_j\right|}\frac{1}{4c^3}\cdot \\
& \cdot\biggl(4ac^2d\Bigl[\mathrm{e}^{c(t-t_j)}+\mathrm{e}^{c(t-t_i)}\Bigr]+4c^3d^2t \\
& \quad -4bcd\mathrm{e}^{c(t-t_i)}\Bigl[c(t-t_i)-1\Bigr]-4bcd\mathrm{e}^{c(t-t_j)}\Bigl[c(t-t_j)-1\Bigr] \\
& \quad +\mathrm{e}^{c(2t-t_i-t_j)}\biggl(2a^2c^2+2abc\Bigl[1+c(t_i+t_j-2t)\Bigr] \\
& \qquad\qquad +b^2\Bigl[1+2c^2(t-t_i)(t-t_j)+c(t_i+t_j-2t)\Bigr]\biggr)\biggr) \quad .
\end{aligned}
$$

Results



The pricing error of a set of coterminal at-the-money swaptions from the constant-weights approximation given by equation (3) and including the shape correction as in equation (16) for a GBP yield curve for August 10th, 2000.

Acknowledgements

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