Investigating Dynamic Dependence Using Copulae

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Abstract

A general methodology for time series modelling is developed which works down from distributional properties to implied structural models including the standard regression relationship. This general to specific approach is important since it can avoid spurious assumptions such as linearity in the form of the dynamic relationship between variables. It is based on splitting the multivariate distribution of a time series into two parts: (i) the marginal unconditional distribution, (ii) the serial dependence encompassed in a general function , the copula. General properties of the class of copula functions that fulfill the necessary requirements for Markov chain construction are exposed. Special cases for the gaussian copula with AR(p) dependence structure and for archimedean copulae are presented. We also develop copula based dynamic dependency measures – auto-concordance in place of autocorrelation. Finally, we provide empirical applications using financial returns and transactions based forex data. Our model encompasses the AR(p) model and allows non-linearity. Moreover, we introduce non-linear time dependence functions that generalize the autocorrelation function.

Keywords: Time Series; Markov chain; Copulas; Likelihood Estimation; Financial Econometrics.

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1 Introduction

The problem of assessing the temporal dependence of financial returns has been an important issue in empirical finance for at least the last three decades, see for instance CAMPBELL AND SHILLER[1988],FAMA AND FRENCH[1989] and ANG AND BEKAERT[2001]. Much of this work has concentrated on linear predictability through linear autocorrelation analysis although much more general dynamic dependency patterns could exist. Copulae provide a general approach to modelling dependence between random variables since they link univariate margins to their full distribution function and in this paper we seek to develop this approach to examining general dynamic dependence and make applications to examine the question of financia lreturn predictability. For an overview of the application of copulae to finance, see EMBRECHTS, MCNEIL and STRAUMANN [1999], BOUYÉ and SALMON [2000] and BOUYÉ, DURRLEMAN, NICKEGHBALI, RIBOULET and RONCALLI [2000].

A time series can be viewed as a single drawing from a multivariate distribution. The goal of this paper is to split this distribution into two components: the margins and the dependence structure given by the Copula. This framework allows to specify *any* univariate distribution for the margins and enables us to consider general non-linear relationships for the time series. The question of the departure from linearity is an important issue quite generally; see TERÄSVIRTA, TJØSTHEIM and GRANGER [1994], and copulae provide a powerful tool to explore this question. PATTON [2001] has also recently explored the use of copulae in time series by studying the dependence between the Deutsche mark - U.S. dollar and Yen - U.S. dollar exchange rate returns. He finds that the dependence pattern is time-varying and asymmetric; a structure that would be difficult to isolate using linear techniques.

In the next section, the concept of a copula is briefly introduced and we demonstrate how a stationary time series - and more generally a stationary Markov chain - can be constructed from a copula function. The multivariate case is briefly discussed and the expression for the transition density function is given. In the third section, we focus on autoregressive (AR) models based on the multivariate gaussian copula to construct stationary Markov processes of $p'th$-order. The dependence structure of a $p'th$-order Markov processes can be captured in an *intrinsic copula* of dimension $(p+1)$ and the dimension of this intrinsic copula provides the minimal representation. So while higher order copulae will capture the same structure non-parsimoniously the intrinsic copula has the lowest dimension required to fully capture the time dependence. Empirically it is frequently difficult to identify the correct dynamic order of a multivariate dynamic system even in the linear case. However in the linear case that order of the intrinsic copula will be directly related the McMillan Degree of the System or the minimal state space representation. The extension to the nonlinear case that we could potentially consider through the use of non-gaussian copulae is as far as we know a completely undeveloped area of research. An alternative class of copulae - archimedean - is then used to construct markov models and some of its properties are given. In the fourth section, we consider the maximum likelihood estimation of time series with a given copula based serial dependence assumption and some discussion of model mispecification is provided. In the fifth section, we apply our model to financial examples and consider auto-concordance measures based on Kendall's Tau and Spearman's rho. Finally we offer some conclusions.

2 Copulas and serial dependence

There are two issues when modelling a one-dimension time series: (i) the choice of the univariate margin and (ii) the time dependence. Joe [1996] proposes a very general way of obtaining stationary time series models with the margins in the convolution-closed infinitely divisible class. He introduces an operator $A(.;\alpha)$ such that for $X \sim F_\theta$, $A(X) \sim F_{\alpha\theta}$ with F_θ such that $\forall(\theta_1,\theta_2) \in \mathbb{R}_+^{*2}, F_{\theta_1} * F_{\theta_2} = F_{\theta_1+\theta_2}$ with $*$ the convolution product. He then constructs a time series as follows: $X_t = A_t(X_{t-1}) + \varepsilon_t$ with $\varepsilon_t \sim IIDF_{(1-\alpha)\theta}$ where the autocorrelation $\alpha \in (0,1)$. Our interest in the current paper will focus on simpler structures such as

$$X_t = g(X_{t-1}, \ldots, X_{t-p}, \epsilon_t)$$

which are implied by the copula that describes the joint density of the data. Models for the conditional higher order moments of the random variables , corresponding to ARCH processes will also be implied by the assumed copula and we take up that question elsewhere[1].

Let us start by defining what we mean by a Copula and some of their properties.

2.1 Definitions

Definition 1 (Nelsen (1998), page 39) *A N-dimensional copula is a function* $\mathbf{C}$ *with the following properties:*

1. Dom $\mathbf{C} = [0,1]^N$;

2. $\mathbf{C}$ *is grounded and N-increasing.*

[1] For additional results on the dependence for stationary Markov chains, we refer the reader to Fang, Hu and Joe [1994] , Hu and Joe [1995]

3. $C_k(u) = u, \forall u \in [0,1], \forall k = 1, \ldots, N$ *with* $C_k(u) = \mathbf{C}(1, \ldots, 1, u, 1, \ldots, 1)$ *the k-th margin of the copula*

Theorem 1 (Sklar's theorem) *Let* $\mathbf{F}$ *be an* N*-dimensional distribution function with continuous margins* $F_1, \ldots, F_N$*. Then* $\mathbf{F}$ *has a unique copula representation:*

$$\mathbf{F}(x_1, \ldots, x_N) = \mathbf{C}(F_1(x_1), \ldots, F_N(x_N)) \tag{1}$$

Let $\mathbf{f}$ be the N-dimensional density function of $\mathbf{F}$.defined as follows:

$$\mathbf{f}(x_1, \ldots, x_N) = \frac{\partial \mathbf{F}(x_1, \ldots, x_N)}{\partial x_1 \cdots \partial x_N} \tag{2}$$

Then we have

$$\mathbf{f}(x_1, \ldots, x_N) = \frac{\partial \mathbf{C}(F_1(x_1), \ldots, F_N(x_N))}{\partial x_1 \cdots \partial x_N}$$

With the notation $u_n = F_n(x_n)$ for $n = 1, \ldots, N$, we obtain

$$\mathbf{f}(x_1, \ldots, x_N) = \frac{\partial \mathbf{C}(u_1, \ldots, u_N)}{\partial u_1 \cdots \partial u_N} \prod_{n=1}^{N} f_n(x_n)$$

with f_n the density corresponding to F_n. The term $\frac{\partial \mathbf{C}(u_1, \ldots, u_N)}{\partial u_1 \cdots \partial u_N}$ is called the copula density of $\mathbf{C}$ and is noted $\mathbf{c}(u_1, \ldots, u_N)$. Obviously,

$$\mathbf{c}(F_1(x_1), \ldots, F_N(x_N)) = \frac{\mathbf{f}(x_1, \ldots, x_N)}{\prod_{n=1}^{N} f_n(x_n)} \tag{3}$$

2.2 Some properties

Our aim initially is simply to construct a model for the conditional expectation of the time series where the serial dependence is implied by the associated copula. However, not all copulae are eligible and some structure must be put on the joint density and hence copula to ensure stationarity. Let assume $\{X_t\}_{t=1\ldots p+1}$ be a stationary time series generated by a p-order Markov process *i.e.*

$$X_t = g(X_{t-1}, \ldots, X_{t-p}, \epsilon_t)$$

for some real-valued function g and ϵ_t ,the innovation which is independent of $\{X_{t-1}, \ldots, X_{t-p}\}$. Let $\mathbf{F} = \mathbf{C}(F, \ldots, F)$ be a $(p+1)$-variate cumulative density function (cdf) with F absolutely continuous. The copula C has to satisfy certain conditions in order to construct a stationary Markov chain and these have been summarised by Joe in the following proposition.

Proposition 2 (Joe (1997), page 245) *A* stationary *Markov chain of order p can be constructed from a $(p+1)$-dimensional copula $\mathbf{C}$ that satisfies the following conditions:*

1. *the bivariate margins $C_{i,j}(u_i, u_j)$ are such that $C_{i,i+l}(u_i, u_{i+l}) = C_{1,1+l}(u_1, u_{1+l})$ for $l = 1, \ldots, p-1$ and $i = 2, \ldots, p+1-l$*

2. *the higher dimensional margins $C_{i_1,\ldots,i_n}(u_1, \ldots, u_n)$ are such that $C_{i_1,\ldots,i_n} = C_{1,i_2-i_1+1,\ldots,i_n-i_1+1}$ for $1 \leq i_1 < \ldots < i_n \leq K$ and $3 \leq n \leq p$*

3. *C is differentiable in its first p arguments*

The two first conditions[2] ensure stability in the dependence structure. Indeed, for a sample $\{x_t\}_{t=1\ldots T}$ the serial dependence has to be the same, for example, between (X_t, X_{t+1}, X_{t+5}) for $t = 1, \ldots, T-5$. The third condition is essentially a technical condition that allows us to compute the density of the process. In short, we see that a time series model with p lags can be deduced from a $(p+1)$-dimensional copula.

2.3 Some properties

The conditional (transition) cdf is:

$$\mathbf{F}(x_t|x_{t-1}, \ldots, x_{t-p}) = \frac{d_1(F(x_{t-p}), \ldots, F(x_t))}{d_2(F(x_{t-p}), \ldots, F(x_{t-1}))} \tag{4}$$

with

$$d_1(u_1, \ldots, u_{p+1}) = \frac{\partial^p \mathbf{C}}{\partial u_1 \ldots \partial u_p}(u_1, \ldots, u_{p+1}) \tag{5}$$

and

$$d_2(u_1, \ldots, u_p) = \frac{\partial^p C_{1,\ldots,p}}{\partial u_1 \ldots \partial u_p}(u_1, \ldots, u_p) \tag{6}$$

where $C_{1,\ldots,p}$ is a p dimensional margin of $\mathbf{C}$ i.e $C_{1,\ldots,p} = \mathbf{C}(u_1, \ldots, u_p, 1)$.

[2] For a 5-dimensional copula, conditions 1 and 2 become

$$\begin{cases} C_{12} = C_{23} = C_{34} = C_{45} \\ C_{13} = C_{24} = C_{35} \\ C_{14} = C_{25} \end{cases}$$

and

$$\begin{cases} C_{123} = C_{234} = C_{345} \\ C_{124} = C_{235} \\ C_{134} = C_{245} \\ C_{1234} = C_{2345} \end{cases}$$

Definition 2 *For a sample* $\{x_t\}_{t=1\ldots T}$ *with copula* $\mathbf{C}(u_1,\ldots,u_T)$ *drawn from a* p'*th-order stationary Markov process , the* ***intrinsic copula*** *is the* ***minimal representation*** *copula with dimension* $(p+1)$ *that encompasses all the dependence structure.*

While this minimal order is unique the representation of the model explaining any moment may not be as is well known from linear time series analysis where, for instance a given $VARMA$ model may be expressed alternatively in a state space form and there are a range of *exchangable* models, see Li and Tsay [1998] and Tiao and Tsay[1989]. It is clear that in the linear context there is a direct relationship between the McMillan degree of the dynamic system and the order of the intrinsic copula. In practice however, since higher order models will non-parsimoniously capture the same dynamic information it is an empirical issue of how to determine the minimal order. While this may be relatively easily achieved in the linear case it is not in the nonlinear dynamic case and the only corresponding work on identifying minimal dynamic orders in nonlinear or chaotic systems we know of is through the correlation dimension of Grassberger and Procaccia [1983]. In fact it may be that the copula approach to this issue is the simplest route to follow in the general case. From Bayes theorem, the conditional density is a function of the copula density

$$\mathbf{f}(x_t|x_{t-1},\ldots,x_{t-p}) = f(x_t)\frac{\mathbf{c}(F(x_{t-p}),\ldots,F(x_t))}{\mathbf{c}(F(x_{t-p}),\ldots,F(x_{t-1}))} \tag{7}$$

The two following properties show that Bayes' theorem provides an elegant way to obtain the copula with the lowest dimension which then captures the general structure of serial dependence within the time series.

Property 1 *For a first-order stationary Markov process , the following relations hold :*

$$\mathbf{c}(u_1,\ldots,u_T) = \prod_{t=2}^{T}\mathbf{c}^*(u_{t-1},u_t)$$

and

$$\mathbf{c}(u_1,\ldots,u_T) = \prod_{t=0}^{\frac{T-k}{k-1}}\mathbf{c}(u_{tk-t+1},u_{(t+1)k-t}) \quad \textit{for} \quad k\geq 2 \quad \textit{and} \quad \binom{T-1}{k-1}\in\mathbb{N} \tag{8}$$

where $\mathbf{c}^*$ *is the intrinsic copula density*

Proof

Note that $\mathbf{f}(x_1,\ldots,x_T) = \mathbf{f}(x_1)\prod_{t=2}^{T}\mathbf{f}(x_t \mid x_{t-1})$ and use equation (7).

For the second property, write $\mathbf{c}(u_{tk-t+1}, u_{(t+1)k-t})$ in terms of the intrinsic copula $\mathbf{c}^*$:

$$\mathbf{c}(u_{tk-t+1}, u_{(t+1)k-t}) = \prod_{i=1}^{k-1} \mathbf{c}^*(u_{t(k-1)+i}, u_{t(k-1)+i+1})$$

Property 2 *For a p-order Markov process, the following relations hold :*

$$\mathbf{c}(u_1, \dots, u_T) = \frac{\prod_{t=p+1}^{T} \mathbf{c}^*(u_{t-p}, \dots, u_t)}{\prod_{t=p+2}^{T} c(u_{t-p}, \dots, u_{t-1})} \tag{9}$$

with $\mathbf{c}^*$ *the intrinsic copula density*

Proof

Note that $\mathbf{f}(x_1, \dots, x_T) = \mathbf{f}(x_1, \dots, x_p) \prod_{t=p+1}^{T} \mathbf{f}(x_t \mid x_{t-1}, \dots, x_{t-p})$ and use equation (7).

Notice that in the case of a p^{th} order Markov processes there will still be relationships between copulae with order greater than the order of the intrinsic copula. Given the intractable form of a general formula, we prefer to give three examples for $p = 2$ that indicate the main intuition. The following shortcut notational is adopted: $\mathbf{c}(u_1, u_2, \dots, u_k) = \mathbf{c}_{12\dots k}$.

Example 1 ***Too much information about serial dependence****. We know about copulae with dimension strictly greater than three such as* $\mathbf{c}_{12345}$ *. Then as*

$$\mathbf{c}_{12345} = \frac{\mathbf{c}^*_{123}\mathbf{c}^*_{234}\mathbf{c}^*_{345}}{c_{23}c_{34}}$$

and $\mathbf{c}_{1234} = \frac{\mathbf{c}^*_{123}\mathbf{c}^*_{234}}{c_{23}}$ *and* $\mathbf{c}_{2345} = \frac{\mathbf{c}^*_{234}\mathbf{c}^*_{345}}{c_{34}}$*, the following relationship arises :*

$$\mathbf{c}^*_{234} = \frac{\mathbf{c}_{1234}\mathbf{c}_{2345}}{\mathbf{c}_{12345}}$$

The dependence structure does not have a minimal representation with a four dimensional copula. However, all the information about serial dependence is available and the three dimensional intrinsic copula can be found.

Example 2 ***Partial information about serial dependence****. We have a knowledge about copulas with dimension strictly lower than three. We have*

$$\mathbf{c}_{1234} = \frac{\mathbf{c}^*_{123}\mathbf{c}^*_{234}}{c_{23}}$$

and extensions of this formul for future periods. All the information about serial dependence is not available and the minimal three dimensional intrinsic copula can not be deduced.

Example 3 ***Full and minimal information about serial dependence.*** *We have a knowledge about copula with dimension three. This is the intrinsic copula and all the information about serial dependence is available.*

3 Gaussian Copula and autoregressive models

In this section, we provide the intuition behind copula functions for linear autoregressive processes. The gaussian AR(1) and AR(p) are explored and their dependence structure is characterized using copulae.Notice that any continuous univariate distribution (not gaussian) will be a candidate to construct a *nonlinear* time series model based on the gaussian copula.

3.1 AR(1) model

Consider the simple AR(1) model:

$$\begin{cases} x_t = c + \phi x_{t-1} + \varepsilon_t \\ \varepsilon_t \sim IID\mathcal{N}(0, \sigma^2) \end{cases} \tag{10}$$

The conditional and unconditional pdfs are well known to be :

$$\begin{cases} x_t | x_{t-1} \sim \mathcal{N}(\phi x_{t-1}, \sigma^2) \\ x_t \sim \mathcal{N}(\frac{c}{1-\phi}, \sigma^2/(1-\phi^2)) \end{cases} \tag{11}$$

A time series sample $\{x_t\}_{t=1...T}$ can be viewed as single draw from $\mathbf{x} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ with density

$$f(\mathbf{x}; \boldsymbol{\mu}, \Sigma) = (2\pi)^{-\frac{T}{2}} \left|\Sigma^{-1}\right|^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-\mathbf{1}}(\mathbf{x}-\boldsymbol{\mu})\right\} \tag{12}$$

where

$$\boldsymbol{\mu} = \mathbf{E}(\mathbf{x}) = \frac{c}{1-\phi} \tag{13}$$

$$\Sigma = \mathbf{E}(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top = \sigma^2 \mathbf{V} = \frac{\sigma^2}{1-\phi^2}\boldsymbol{\rho} \tag{14}$$

with

$$\mathbf{V} = \frac{1}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{T-1} \\ \phi & 1 & \phi & \cdots & \phi^{T-2} \\ \phi^2 & \phi & 1 & \cdots & \phi^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{T-1} & \phi^{T-2} & \phi^{T-3} & & 1 \end{pmatrix} \tag{15}$$

Proposition 3 *The copula density function corresponding to time series $\{x_t\}_{t=1\ldots T}$ from an AR(1) process is*

$$\mathbf{c}\left(u_1,\ldots,u_t,\ldots,u_T;\boldsymbol{\rho}\right)=(1-\phi^2)^{\frac{1-T}{2}}\exp\left(-\frac{1}{2}\boldsymbol{\varsigma}^\top\left(\boldsymbol{\rho}^{-1}-\mathbb{I}\right)\boldsymbol{\varsigma}\right) \tag{16}$$

with $\boldsymbol{\varsigma}_t=\Phi^{-1}\left(u_t\right)$.

Proof

We have $\prod_{t=1}^{T} f_t\left(x_t\right)=(2\pi)^{-\frac{T}{2}}\sigma^{-T}(1-\phi^2)^{\frac{T}{2}}\exp\left\{-\frac{1}{2}\boldsymbol{\varsigma}^\top\boldsymbol{\varsigma}\right\}$ with $\boldsymbol{\varsigma}_t=\Phi^{-1}\left(u_t\right)$. From (3), the copula density is obtained. As $\mathbf{V}^{-1}=L^\top L$ with L a lower triangular matrix with diagonal product $\sqrt{1-\phi^2}$, $\left|\mathbf{V}^{-1}\right|=1-\phi^2$. Then note that $(1-\phi^2)\boldsymbol{\rho}^{-1}=\mathbf{V}^{-1}$.

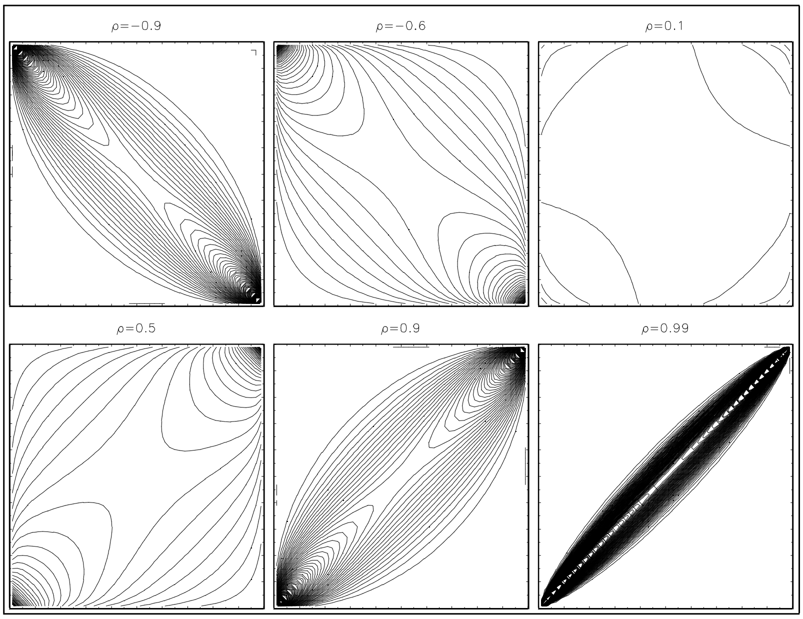

Figure 1: Contour plot of the bivariate gaussian copula with different values for the serial dependence parameter $\rho=-0.9,-0.6,-0.1,0.5,0.9,0.99$.

3.2 AR(p) model

The previous results can be extended to the linear AR(p) case:

$$\left\{\begin{array}{l} x_t=c+\phi_1 x_{t-1}+\phi_2 x_{t-2}+\ldots+\phi_p x_{t-p}+\varepsilon_t \\ \varepsilon_t\sim IID\mathcal{N}(0,\sigma^2)\end{array}\right. \tag{17}$$

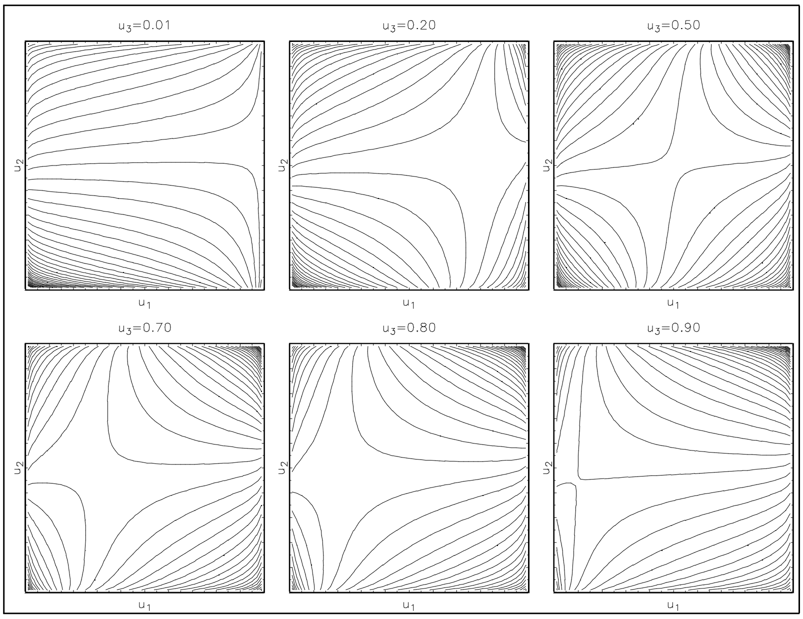

Figure 2: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.1$

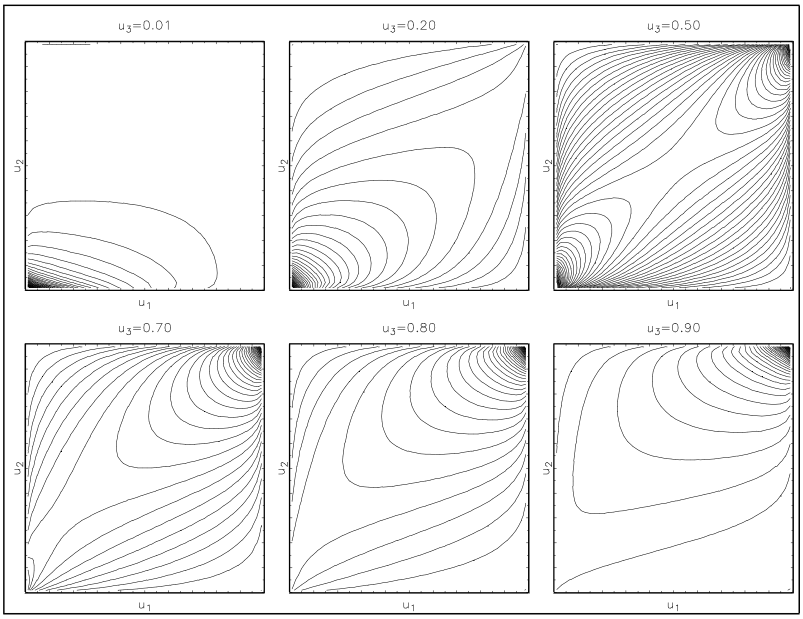

Figure 3: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.5$

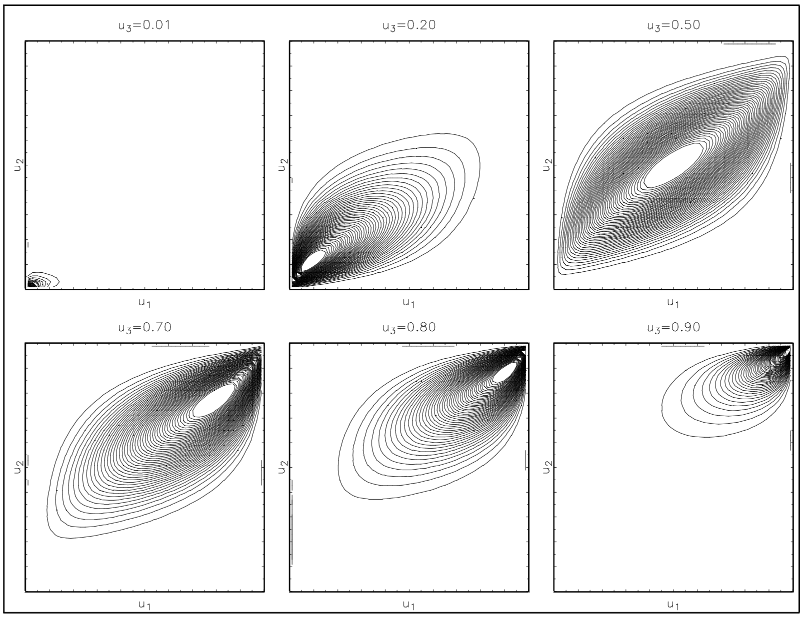

Figure 4: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.9$

A time series sample $\{x_t\}_{t=1\ldots T}$ can be viewed as single draw from $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with

$$\Sigma = \frac{\sigma^2}{1 - \sum_{j=1}^{p} \rho_j \phi_j} \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_T \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T-1} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{T-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_T & \rho_{T-1} & \rho_{T-2} & & 1 \end{pmatrix} \tag{18}$$

with ρ_j the autocorrelations[3] that fulfill the Yule-Walker equations:

$$\begin{cases} \rho_0 = 1 \\ \rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} + \ldots + \phi_p \rho_{j-p} \quad \text{for} \quad j = 1, 2, \ldots \end{cases} \tag{19}$$

Property 3 *The AR(p) gaussian intrinsic copula density function is given by*

$$\mathbf{c}^* \left(u_{t-p}, \ldots, u_t; \boldsymbol{\rho}^*\right) = \frac{1}{|\boldsymbol{\rho}^*|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \boldsymbol{\varsigma}^\top \left(\boldsymbol{\rho}^{*-1} - \mathbb{I}\right) \boldsymbol{\varsigma}\right) \tag{20}$$

[3] The autocorrelations $(\rho_0, \ldots, \rho_{p-1})$ are obtained by taking the first p elements of the first column of the matrix $\left(\mathbb{I}_{p^2} - C \quad C\right)^{-1}$ where

$$C = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\textit{with } \boldsymbol{\rho}^* = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_p \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_p & \rho_{p-1} & \rho_{p-2} & & 1 \end{pmatrix}$$

3.3 An alternative dependence model: the archimedean class

Another famous class of copula functions, the Archimedean class, provides an alternative to the gaussian copula. This class is very useful since each member of this class can be characterised by a simple generator function. However, its extension from the bivariate to the multivariate case often becomes intractable. Indeed, since each correlation parameter of a gaussian copula provides information about the dependence between each pair of random variables so for N variables, there are $N(N-1)/2$ parameters. For Archimedean copulae, the dependence is characterised by only $(N-1)$ parameters. We shall focus on the bivariate case i.e. on first-order Markov processes. Genest and MacKay [1996] provided a definition of this family:

$$\mathbf{C}(u_1, \ldots, u_N) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_N)) \tag{21}$$

with $\varphi(u)$ a C^2 function with $\varphi(1) = 0$, $\varphi'(u) < 0$ and $\varphi''(u) > 0$ for all $0 \leq u \leq 1$. One technique by which to construct multivariate archimedean copulae is the compound method where

$$\begin{array}{l} \mathbf{C}(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2)) \\ \mathbf{C}(u_1, u_2, u_3) = \mathbf{C}(\mathbf{C}(u_1, u_2), u_3) \\ \vdots \\ \mathbf{C}(u_1, \ldots, u_N) = \mathbf{C}(\mathbf{C}(u_1, , \ldots, u_{N-1}), u_N) \end{array} \tag{22}$$

The function $\varphi(u)$ is called the *generator* of the copula and essentially identifies the copula function. Genest and Rivest [1993] propose a method to identify an Archimedean copula by comparing the true value of a function $\lambda(u)$ to its nonparametric estimate, where

$$\begin{aligned} \lambda(u) &= u - \Pr\{\mathbf{C}(U_1, \ldots, U_N) \leq u\} \\ &= \sum_{n=1}^{N} (-1)^{n-1} \frac{\varphi^n(u)}{n!} \omega_{n-1}(u) \end{aligned} \tag{23}$$

with

$$\begin{cases} \omega_0(u) = (\varphi\prime(u))^{-1} \\ \omega_n(u) = (\varphi'(u))^{-1} \left(\frac{\partial\, \omega_{n-1}(u)}{\partial\, u} \right) \end{cases}$$

Copula	$\mathbf{C}(u_1,u_2)$	$\varphi(u)$	$\delta\in$	dependence
$\mathbf{C}^{\perp}$	u_1u_2	$-\ln u$		
Gumbel	$\exp\left(-\left((-\ln u_1)^{\delta}+(-\ln u_2)^{\delta}\right)^{\frac{1}{\delta}}\right)$	$(-\ln u)^{\delta}$	$[1,\infty)$	+
Joe	$1-\left((1-u_1)^{\delta}+(1-u_2)^{\delta}-(1-u_1)^{\delta}(1-u_2)^{\delta}\right)^{\frac{1}{\delta}}$	$-\ln\left(1-(1-u)^{\delta}\right)$	$[1,\infty)$	+
Frank	$-\frac{1}{\delta}\ln\left(1+\frac{(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)}{e^{-\delta}-1}\right)$	$-\ln\frac{e^{-\delta u}-1}{e^{-\delta}-1}$	$\mathbb{R}^*$	+ and -

Table 1: Three famous bivariate archimedean copulae $\mathbf{C}(u_1,u_2)$ with the generator function $\varphi(u)$ and properties

We refer to Barbe, Genest, Ghoudi and Rémillard [1996] for the proof. A non parametric estimate of $\lambda(u)$ is given by

$$\hat{\lambda}(u) = u - \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}_{[\vartheta_i \le u]} \tag{24}$$

with

$$\vartheta_i = \frac{1}{T-1}\sum_{t=1}^{T}\mathbf{1}_{[x_1^t<x_1^i,\dots,x_N^t<x_N^i]}$$

4 Maximum likelihood estimation

Let $(\boldsymbol{\gamma},\boldsymbol{\delta})$ be the vector of parameters to be estimated and (Γ,Δ) the parameter spaces where $\boldsymbol{\gamma}$ characterizes the margins and $\boldsymbol{\delta}$ the dependence. For a sample of size T, the log-likelihood function $\ell_t(\boldsymbol{\gamma},\boldsymbol{\delta})$ can be constructed so that $(\hat{\gamma}_{\mathrm{ML}},\hat{\delta}_{\mathrm{ML}})$ is the Maximum Likelihood (ML) estimator given by

$$(\hat{\boldsymbol{\gamma}}_{\mathrm{ML}},\hat{\boldsymbol{\delta}}_{\mathrm{ML}}) = \underset{(\boldsymbol{\gamma},\boldsymbol{\delta})\in(\Gamma,\Delta)}{\mathrm{ArgMax}}\sum_{t=1}^{T}\ell_t(\boldsymbol{\gamma},\boldsymbol{\delta}) \tag{25}$$

with asymptotic normality:

$$(\hat{\boldsymbol{\gamma}}_{\mathrm{ML}},\hat{\boldsymbol{\delta}}_{\mathrm{ML}}) \longrightarrow T^{-\frac{1}{2}}\mathcal{N}\left((\boldsymbol{\gamma}_0,\boldsymbol{\delta}_0),\mathcal{I}^{-1}(\boldsymbol{\gamma}_0,\boldsymbol{\delta}_0)\right) \tag{26}$$

with $\mathcal{I}(\boldsymbol{\gamma}_0,\boldsymbol{\delta}_0)$ the Fisher information matrix.

The likelihood for a sample $\{x_t\}_{t=1\dots T}$ of a p-order Markov process can be deduced from (7):

$$\begin{aligned} L(x_1,\dots,x_T;\boldsymbol{\gamma},\boldsymbol{\delta}) &= \mathbf{f}(x_1,\dots,x_T;\boldsymbol{\gamma},\boldsymbol{\delta}) \\ &= \prod_{t=1}^{T} f(x_t;\boldsymbol{\gamma})\frac{\prod_{t=p+1}^{T}\mathbf{c}^*(F(x_{t-p};\boldsymbol{\gamma}),\dots,F(x_t;\boldsymbol{\gamma});\boldsymbol{\delta})}{\prod_{t=p+2}^{T}c(F(x_{t-p};\boldsymbol{\gamma}),\dots,F(x_{t-1};\boldsymbol{\gamma});\boldsymbol{\delta})} \end{aligned} \tag{27}$$

and the log-likelihood estimator is then given by

$$\begin{aligned}\ell(x_1,\ldots,x_T;\boldsymbol{\gamma},\boldsymbol{\delta}) &= \ln \mathbf{f}(x_1,\ldots,x_T;\boldsymbol{\gamma},\boldsymbol{\delta}) \\ &= \sum_{t=1}^{T}\ln f(x_t;\boldsymbol{\gamma}) + \sum_{t=p+1}^{T}\ln \mathbf{c}^*(F(x_{t-p};\boldsymbol{\gamma}),\ldots,F(x_t;\gamma);\boldsymbol{\delta}) \\ &\quad - \sum_{t=p+2}^{T}\ln c(F(x_{t-p};\boldsymbol{\gamma}),\ldots,F(x_{t-1};\boldsymbol{\gamma});\boldsymbol{\delta})\end{aligned} \tag{28}$$

In the case of the gaussian copula, we have

$$\ell(\boldsymbol{\gamma},\boldsymbol{\rho}^*) = \sum_{t=1}^{T}\ln f(x_t;\boldsymbol{\gamma}) - \frac{T-p}{2}\ln|\boldsymbol{\rho}^*| - \frac{1}{2}\sum_{t=p+1}^{T}\varsigma_t^\top\left(\boldsymbol{\rho}^{*-1}-\mathbb{I}\right)\varsigma_t - \frac{T-p-1}{2}\ln|\boldsymbol{\rho}| - \frac{1}{2}\sum_{t=p+2}^{T}\varsigma_{t-1}^\top\left(\boldsymbol{\rho}^{-1} - \right. \tag{29}$$

with

$$\begin{cases} Rank(\boldsymbol{\rho}^*) = p+1 \\ Rank(\boldsymbol{\rho}) = p \\ \varsigma_t = \left(\Phi^{-1}\left(F(x_{t-p})\right),\ldots,\Phi^{-1}\left(F(x_t)\right)\right) \\ \varsigma_{t-1} = \left(\Phi^{-1}\left(F(x_{t-p})\right),\ldots,\Phi^{-1}\left(F(x_{t-1})\right)\right)\end{cases}$$

and the ML estimate of $\boldsymbol{\rho}$ is

$$\hat{\boldsymbol{\rho}}^*_{\mathrm{ML}} = \frac{1}{T}\sum_{t=1}^{T}\varsigma_t^\top\varsigma_t \tag{30}$$

As noted in Joe and Xu [1996] and Bouyé, Durrleman, Nickeghbali, Riboulet and Roncalli [2000], three maximum likelihood methods are available.

1. *The Exact Maximum Likelihood (EML) method:* Parameters of the copula and marginals are estimated simultaneously. The time series sample, $\mathbf{x} = \{x_t\}_{t=1\ldots T}$, has density

   $$f(\mathbf{x};\boldsymbol{\gamma},\boldsymbol{\delta}) = c\left(F(x_1;\boldsymbol{\gamma}),\ldots,F(x_T;\boldsymbol{\gamma}),\boldsymbol{\delta}\right)\prod_{t=1}^{T}f(x_t;\boldsymbol{\gamma})$$

   The log-likelihood of the joint distribution function for a sample of size T is

   $$L(\boldsymbol{\gamma},\boldsymbol{\delta}) = \sum_{t=1}^{T}\log f(\mathbf{x}_t;\boldsymbol{\gamma},\boldsymbol{\delta}) \tag{31}$$

   The MLE estimates $(\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{\delta}})$ maximize L, are obtained by solving $\left(\frac{\partial L}{\partial \gamma},\frac{\partial L}{\partial \delta}\right)^\top = \mathbf{0}$.

2. *The Inference Function for Margins (IFM) method* is a two-step procedure. First, parameters of the marginals are estimated. Second, MLE is applied to estimate the dependence parameters of the copula. The log-likelihood functions for the univariate margin L_m is considered:

$$L_m(\boldsymbol{\gamma}) = \sum_{t=1}^{T} \log f(x_t; \boldsymbol{\gamma}) \tag{32}$$

The estimates $\tilde{\boldsymbol{\gamma}}$ maximize L_m. The log-likelihood of the joint distribution function $L(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\delta})$ is maximized over $\boldsymbol{\delta}$ to obtain $\tilde{\boldsymbol{\delta}}$. Finally, the IFM estimates $(\tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}})$ are obtained by solving $\left(\frac{\partial L_m}{\partial \gamma}, \frac{\partial L}{\partial \delta}\right) = \mathbf{0}$.

3. *The Canonical Maximum Likelihood (CML) method:* Only the parameters of the copula are estimated. The empirical cdfs are obtained by mapping variables to uniforms. The margins are mapped to uniforms:

$$\mathbf{X} \in \mathbf{R}^T \longmapsto \mathbf{u} \in [0,1]^T$$

The parameters of the copula $\boldsymbol{\delta}$ are obtained by maximizing the log-likelihood of copula cdf

$$L_c(\boldsymbol{\delta}) = \sum_{t=1}^{T} \log c(\mathbf{u}_t; \boldsymbol{\delta}) \tag{33}$$

The estimate $\bar{\boldsymbol{\delta}}$ is obtained from solving $\frac{\partial L_c}{\partial \boldsymbol{\delta}} = \mathbf{0}$.

5 Financial applications

We consider the annualized daily log-returns for five indices: Cazenove small companies (CAZSCOS), Barings (BARINGS), S&P 500 (SP500), Nasdaq 100 (NASDAQ) and MSCI Singapore (MSSING), from January 1983 to March 2000. The sample size is $T = 4499$. A number of condidate marginal distributions were tested (Gaussian, Weibull, Student and Burr3), but the Burr3was found to the each of the series best. This is shown by the Kolmogorov-Smirnov statistics (KS) in Appendix A. The Burr3 has the following distribution :

$$F(x; \boldsymbol{\gamma}) = \left[1 - \frac{1}{1 + (x/\tau)^{\alpha}}\right]^{\lambda} \quad \text{with} \quad x \in \mathbb{R}^+ \tag{34}$$

with the parameters $\boldsymbol{\gamma} = (\alpha, \lambda, \tau)$. The probability density function (pdf) is

$$f(x; \boldsymbol{\gamma}) = \frac{\alpha \lambda x^{\alpha\lambda - 1} \tau^{\alpha}}{(x^{\alpha} + \tau^{\alpha})^{\lambda+1}} \tag{35}$$

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
$\delta=\rho$	0.383	0.206	0.034	0.089	0.188
	(0.012)	*(0.014)*	*(0.015)*	*(0.015)*	*(0.014)*
LogLik	-7041.3	-8983.6	-9739.8	-11541.7	-11016.7
GUMBEL					
δ	1.352	1.145	1.030	1.076	1.154
	(0.016)	*(0.012)*	*(0.009)*	*(0.011)*	*(0.012)*
LogLik	-6973.6	-8971.6	-9733.1	-11523.4	-10977.5
JOE					
δ	1.448	1.173	1.039	1.095	1.194
	(0.023)	*(0.017)*	*(0.012)*	*(0.015)*	*(0.018)*
LogLik	-7059.5	-8998.5	-9733.3	-11529.6	-10999.5
FRANK					
δ	2.662	1.266	0.154	0.687	1.206
	(0.099)	*(0.093)*	*(0.093)*	*(0.095)*	*(0.095)*
LogLik	-7022.1	-8987.1	-9740.8	-11532.0	-11015.6
INDEP					
LogLik	-7385.7	-9078.9	-9742.2	-11558.3	-11096.3

Table 2: IFM estimates for various copulae under the assumption of a first order Markov process.

The distribution can be split into two parts : positive and negative returns and the corresponding parameters are superscripted $+$ or $-$ depending on the side of the estimated distribution. The maximum likelihood estimates and the Kolmogorov-Smirnov (KS) values[4] for the selected distributions are reported in Table A1 of the Appendix. We estimated the parameters using CML method for the Gaussian AR(1) copula and the three archimedean copulae described in Table 1. The estimated values are given in Table 2. Figures in Appendix plot the non-parametric estimator $\hat{\lambda}(u)$ with the fitted ML values of the independent and archimedean copulae.

The Gumbel copula clearly best fits the dependence of the data for illiquid markets such as for CAZSCOS, BARINGS and MSSING. Not surprisingly, the SP500 is the more liquid market which is also best fit by the Gumbel[5]. Looking at the 95% confidence intervals, we can see that the hypothesis of serial independence can only be rejected for CAZSCOS.

[4] *, **, *** mean that the null hypothesis of the Kolmogorov-Smirnov test (the true distribution equals the estimated one) is respectively rejected at 1%, 5% and 10% level.

[5] Notice that a simple model selection can be based on the likelihood values themselves in this case since there is a single parameter in each of the separate families and so comparison by the likelihood value corresponds to selection by Akaike's Information Criterion (AIC).

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
$\delta = \rho$	0.379	0.206	0.031	0.082	0.192
	(0.012)	*(0.014)*	*(0.015)*	*(0.015)*	*(0.014)*
LogLik	346.2	96.9	2.2	15.0	83.5
GUMBEL					
δ	1.345	1.143	1.027	1.070	1.153
	(0.015)	*(0.012)*	*(0.008)*	*(0.010)*	*(0.012)*
LogLik	414.7	107.9	9.7	33.3	120.2
JOE					
δ	1.441	1.169	1.034	1.084	1.191
	(0.023)	*(0.017)*	*(0.011)*	*(0.015)*	*(0.017)*
LogLik	330.0	80.3	9.5	27.0	96.8
FRANK					
δ	2.604	1.259	0.142	0.654	1.211
	(0.097)	*(0.093)*	*(0.092)*	*(0.092)*	*(0.094)*
LogLik	359.5	92.2	1.2	25.1	82.7

Table 3: CML estimates for various copulae under the assumption of a first order Markov process.

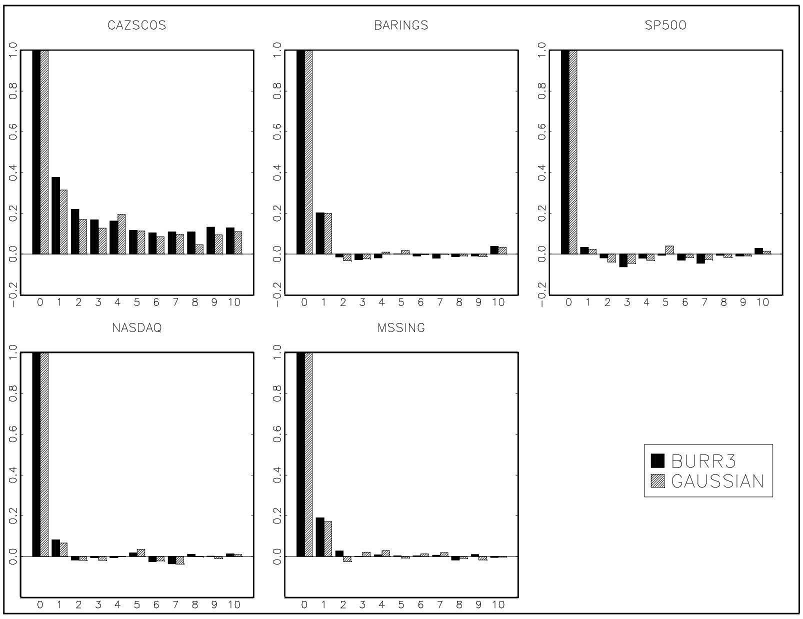

Figure 5: Empirical autocorrelation function under alternative hypothesis about margins : Burr3 and Gaussian.

One main advantage of using copulae is that all the dependence structure is captured by the copula function itself since it is obviously not held in the marginal distributions. Measures of dependence based on the copula have the advantage that they are also invariant to monotonic transformations of the data. Auto-correlation analysis as an approach to measuring dynamic dependence suffers from the same serious limitations that restrict the use of correlation as a measure of association. In particular the autocorrelogram is designed to detect only linear autoregressive processes. It is therefore natural to consider extending methods of detecting potentially nonlinear dynamic structure to copula based measures of dependence that will be applicable outside the class of elliptic distributions such as the Gaussian. Several alternative measures of dependence immediately suggest themselves; in particular auto-concordance measures as opposed to autocorrelation. Two measures of concordance are given by Kendall's Tau and Spearman's rho which may be defined in general in terms of the parameters of the copula. To define the auto-concordance coefficients we treat the original variable and its lag as the two random variables in what follows. The two concordance measures can be used: so the Kendall p-order auto-concordance coefficient is

$$\tau_C\left(p\right) = 4\int\int_{[0,1]^2} C(u_t, u_{t-p})\mathbf{d}C(u_t, u_{t-p}) - 1 \tag{36}$$

and the Spearman rank p-order auto-concordance coefficient is

$$\rho_C\left(p\right) = 12\int\int_{[0,1]^2} \left(C(u_t, u_{t-p}) - uv\right) du_t du_{t-p} \tag{37}$$

Spearman's rank correlation coefficient is essentially the ordinary correlation of $\rho(F_1(X_1), F_2(X_2))$ for two random variables $X_1 \backsim F_1(.)$ and $X_2 \backsim F_2(.)$. Notice the explicit comparison with the product copula ,uv,representing independence. Essentially these two measures of concordance measure the degree of *monotonic* dependence as opposed to the Pearson Correlation which measures the degree of *linear* dependence. Both achieve a value of unity for the bivariate Fréchet upper bound where one variable is a strictly increasing transformation of the other and minus one for the Fréchet lower bound (one variable is a strictly decreasing transform of the other).

Essentially using these auto-concordance measures should enable us to detect monotonic but non-linear dynamic dependence in non-gaussian assets and hence would appear to be useful in financial applications. Further measures of copula based dynamic dependence could be based on dynamic tail area dependency measures.

Lag	tau	rho	acf
1	0.26	0.37	0.32
2	0.15	0.22	0.17
3	0.11	0.17	0.12
4	0.11	0.16	0.19
5	0.09	0.13	0.11
6	0.08	0.11	0.09
7	0.08	0.12	0.10
8	0.08	0.12	0.05
9	0.09	0.14	0.09
10	0.09	0.13	0.11
11	0.06	0.10	0.12
12	0.06	0.09	0.04

All entries significant

Table 4: Auto-concordance / Autocorrelation CAZSCOS

Lag	tau	rho	acf
1	0.13418*	0.196*	0.201*
2	-0.00163	-0.003	-0.0286
3	-0.0235*	-0.035*	-0.0223
4	-0.0184*	-0.027*	0.0109
5	0.00138	0.0024	0.0157
6	-0.0059	-0.009	-0.0019
7	-0.0131	-0.019	-0.0013
8	-0.008	-0.0127	-0.0084
9	-0.0115	-0.017	-0.0138
10	-0.0237*	0.035*	0.0340*
11	-0.006	-0.009	0.0111
12	-0.004	-0.006	0.0034

Table 5: Auto-concordance / Autocorrelation BARINGS

The following tables compare the auto-concordance and auto-correlation for the return series described above[6].

The general conclusion we can draw from these results is that within the same general pattern of dependence some potentially important differences emerge between the auto-concordance and auto-correlation coefficients. The same general dependence structure is indicated by both the auto-concordance measures. The distributions of these return series show the classic pattern of relatively

[6]Star's(*) indicate values significantly different from zero at a 5% level.

Lag	tau	rho	acf
1	0.071*	0.102*	0.0656*
2	-0.008	-0.012	-0.0178
3	-0.0019	-0.003	-0.0194
4	-0.003	-0.004	0.0004
5	0.009	0.014	0.0339*
6	-0.005	-0.0082	-0.0213
7	-0.022*	-0.032*	-0.0372*
8	0.0087	0.0127	-0.0016
9	0.0054	0.007	-0.0112
10	0.007	0.01	0.0096
11	-0.0018	-0.003	-0.0089
12	0.018	0.027	0.0232

Table 6: Auto-concordance / Autocorrelation NASDAQ

Lag	tau	rho	acf
1	0.0155	0.023	0.0229
2	-0.012	-0.018	-0.0372*
3	-0.043*	-0.063*	-0.0459*
4	-0.0145	-0.021	-0.0295*
5	-0.0082	-0.012	0.0378*
6	-0.0173	-0.025	-0.0163
7	-0.027*	-0.040*	-0.0273
8	-0.0029	-0.004	-0.0162
9	-0.011	-0.016	-0.0099
10	0.0144	0.022	0.0141
11	0.0003	0.005	-0.0063
12	0.0179	0.026	0.0066

Table 7: Auto-concordance / Autocorrelation SP500

Lag	tau	rho	acf
1	0.1266*	0.182*	0.1721*
2	0.0208*	0.030*	-0.0237
3	0.004	0.005	0.0217
4	0.0032	0.005	0.0301*
5	0.008	0.011	-0.0070
6	0.003	0.005	-0.0123
7	0.005	0.0082	0.0187
8	-0.016	-0.022	-0.0096
9	0.0077	0.011	-0.0162
10	-0.0027	-0.004	-0.0032
11	0.019*	0.028	0.0274
12	0.031*	0.046*	0.0454*

Table 8: Auto-concordance / Autocorrelation MSSING

Lag	tau	rho	acf
1	0.171	0.252	0.259
2	0.137	0.203	0.206
3	0.120	0.179	0.322
4	0.11	0.165	0.294
5	0.096	0.144	0.171
6	0.102	0.152	0.212
7	0.106	0.158	0.338
8	0.084	0.126	0.185
9	0.086	0.129	0.165
10	0.108	0.162	0.259
11	0.090	0.135	0.273
12	0.094	0.139	0.179

Table 9: Auto-concordance / Autocorrelation DM2000-2 Transactions

small skewness but substantial excess kurtosis. The relative symmetry of these distributions may well explain the lack of any dramatic difference being indicated between the auto-concordance and the auto-correlation coefficients. In order to investigate this further we have applied the same procedures to two duration series drawn from a sample of all transactions from the DM2000-2 electronic order book screen trading system for the Dollar DeutscheMark[7]. We consider the question of dynamic dependence within the duration between transactions and also the order flow duration onto the DM2000-2 system and since these must be non-negative their distributions must be asymmetric and lie entirely in the positive quadrant. In fact they are relatively well represented by members of the Weibull distribution. The following two tables provide the auto-concordance and auto-correlation coefficients for these two series of 26578 observations (order entries) and 4404 (transactions).

Unfortunately we see little discrimination between the measures in these last two tables since all entries are significantly different from zero and show essentially the same pattern. We clearly need to find more subtle examples in order to demonstrate the value of the auto-concordance functions. However this does not imply that other dynamic dependency patterns may be discovered using Copulae. The most obvious choice would seem to be looking at dependency and the dynamic evolution in the tails of the distribution of returns.

[7] Further details of this data set and an analysis of its structure can be found in Hillman and Salmon (2000)

Lag	**tau**	**rho**	**acf**
1	0.160	0.237	0.267
2	0.157	0.223	0.277
3	0.155	0.230	0.251
4	0.148	0.220	0.234
5	0.142	0.221	0.215
6	0.137	0.203	0.205
7	0.139	0.207	0.250
8	0.137	0.203	0.234
9	0.126	0.187	0.206
10	0.133	0.198	0.212
11	0.124	0.184	0.180
12	0.127	0.190	0.188

Table 10: Auto-concordance / Autocorrelation DM2000-2 Order flow entries

6 Conclusion

We have taken the first steps in this paper to develop an empirical methodology to investigate the dynamic dependence in non-gaussian time series and financial returns in particular using copula functions. Some properties of the class of copula functions that allow us to construct p-order markov processes have been presented and it is important to note that any density can be assumed for the margins.We intend to extend this work to investigate the multivariate predictability issue of returns using variables such as dividend yields, earnings and interest rates which would then enable us to determine how far the lack of independence of returns can be accounted for by nonlinear forms of dynamic dependence. It is conceptually easy to move from considering the conditional expectation derived from the copula to consider the implied model of conditional volatility and its dynamic strucutre. This is again work we are following up elsewhere. A critical assumption of this paper is that the density of the margins does not change through time. One further extension would be to look at possible regime changes both for the marginal density and the dependence structure itself.

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A ML Estimates for margins

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
μ	0.088	0.093	0.142	0.219	0.068
	(0.011)	*(0.010)*	*(0.010)*	*(0.009)*	*(0.009)*
σ	1.727	2.031	2.616	3.768	3.614
	(0.018)	*(0.021)*	*(0.028)*	*(0.041)*	*(0.038)*
LogLik	-8603.2	-9515.6	-10370.1	-11865.4	-12032.2
KS-Test	0.118*	0.058*	0.078*	0.058*	0.096*
WEIBULL					
a^+	1.042	1.104	1.070	1.134	0.985
	(0.014)	*(0.017)*	*(0.016)*	*(0.015)*	*(0.014)*
x^+	0.992	1.488	1.795	2.797	2.237
	(0.019)	*(0.026)*	*(0.036)*	*(0.049)*	*(0.047)*
a^-	0.882	1.029	0.990	1.048	0.932
	(0.014)	*(0.016)*	*(0.015)*	*(0.018)*	*(0.014)*
x^-	1.047	1.416	1.726	2.806	2.200
	(0.028)	*(0.030)*	*(0.038)*	*(0.062)*	*(0.047)*
p	0.577	0.527	0.539	0.548	0.518
	(0.008)	*(0.007)*	*(0.007)*	*(0.008)*	*(0.008)*
LogLik	-7475.6	-9107.8	-9770.4	-11569.1	-11159.5
KS-Test	0.021**	0.011	0.013	0.011	0.015
STUDENT					
ν	3.165	2.119	1.561	0.982	1.172
	(0.126)	*(0.068)*	*(0.043)*	*(0.022)*	*(0.028)*
LogLik	-7471.2	-9247.9	-10040.3	-12529.6	-11563.5
KS-Test	0.084*	0.076*	0.098*	0.161*	0.099*
BURR3					
α^+	2.535	3.082	3.050	2.946	2.600
	(0.096)	*(0.131)*	*(0.139)*	*(0.129)*	*(0.112)*
λ^+	0.404	0.303	0.291	0.340	0.343
	(0.024)	*(0.019)*	*(0.020)*	*(0.021)*	*(0.022)*
τ^+	1.267	2.263	2.845	3.945	3.275
	(0.050)	*(0.078)*	*(0.104)*	*(0.143)*	*(0.134)*
α^-	2.173	2.944	2.469	2.784	2.473
	(0.094)	*(0.132)*	*(0.112)*	*(0.130)*	*(0.103)*
λ^-	0.415	0.306	0.390	0.336	0.354
	(0.029)	*(0.021)*	*(0.028)*	*(0.025)*	*(0.022)*
τ^-	1.358	2.179	2.306	4.081	3.166
	(0.073)	*(0.084)*	*(0.111)*	*(0.171)*	*(0.134)*
p	0.577	0.527	0.539	0.548	0.518
	(0.008)	*(0.007)*	*(0.008)*	*(0.007)*	*(0.008)*
LogLik	-7385.7	-9078.9	-9742.2	-11558.3	-11096.3
KS-Test	0.006	0.007	0.013	0.010	0.006

Table A1 - ML Estimates for margins.

B Inverse of the Burr3 distribution

The Burr3 distribution :

$$G(x;\boldsymbol{\gamma}) = \left[1 - \frac{1}{1+(x/\tau)^{\alpha}}\right]^{\lambda} = u_G \quad \text{with} \quad x \in \mathbb{R}^{+}$$

with the parameters $\boldsymbol{\gamma} = (\alpha, \lambda, \tau)$. Then it comes that its inverse $G^{[-1]}$ is

$$G^{[-1]}(u_G;\boldsymbol{\gamma}) = \tau \left[\frac{1}{1-u_G^{1/\lambda}} - 1\right]^{1/\alpha} \quad \text{with} \quad u_G \in [0,1] \tag{38}$$

A distribution F for positive and negative values is constructed:

$$F\left(x;\boldsymbol{\gamma}^{-},\boldsymbol{\gamma}^{+}\right) = 1 - p - \mathbb{I}_{\{x<0\}}\left(1-p\right)G(|x|\,;\boldsymbol{\gamma}^{-}) + \mathbb{I}_{\{x\geq 0\}}pG(x;\boldsymbol{\gamma}^{+}) = u_F$$

Then, the quasi-inverse is slightly modified, depending on the sign of x:

$$\begin{aligned} x &< 0,\ u_G = \frac{u_F}{1-p} - 1 \\ x &\geq 0,\ u_G = (1-u_F)\,(1-p) \end{aligned}$$

C Gaussian copula based simulations

We may postulate other distributions for the margins with a dependence structure summarized by a gaussian copula with AR(p) structure correlation matrix. To illustrate this idea, we simulate two times series of size $T = 200$ with an AR(1) intrinsic gaussian copula but with different univariate margins (standard normal and Student) - see Figure 6. A simulation for an AR(2) intrinsic gaussian copula is also reported in Figure 7.

D Additional figures

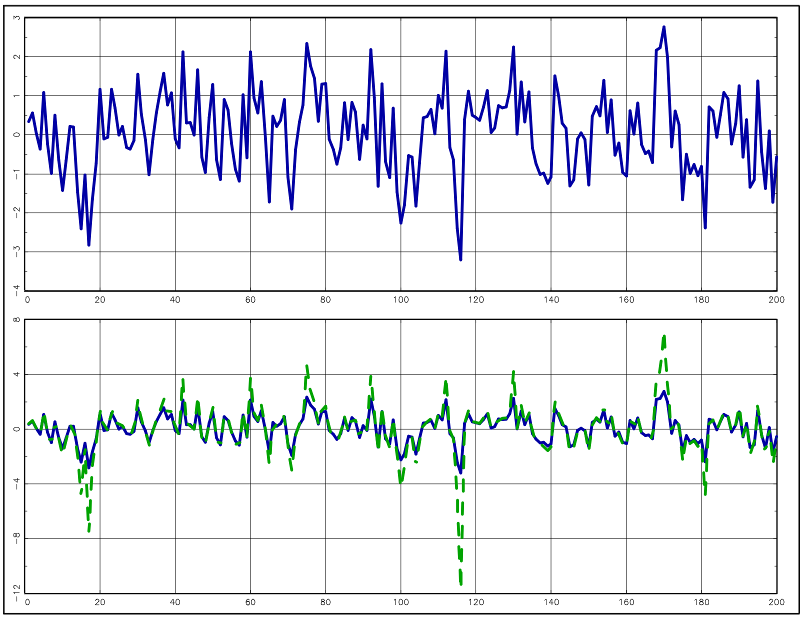

Figure 6: Simulation of two times series with the same intrinsic gaussian copula with an AR(1) matrix where $\phi = 0.3$. The marginal distributions are different: the solid line corresponds to standard normal, the dashed line to Student with $\nu = 3$ degrees of freedom.

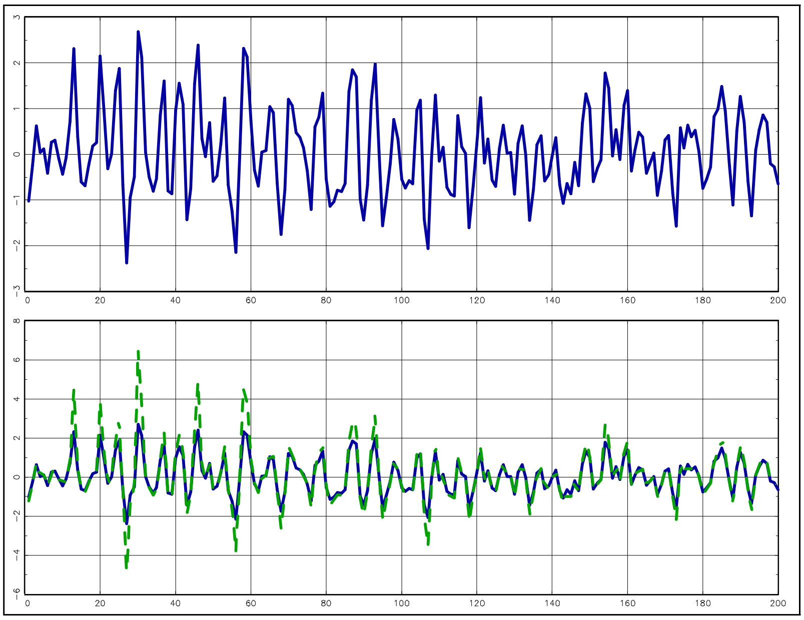

Figure 7: Simulation of two times series with the same intrinsic gaussian copula with an AR(2) matrix where $(\phi_1, \phi_2) = (0.6, -0.5)$. The marginal distributions are different: the solid line corresponds to standard normal, the dashed line to Student with $\nu = 3$ degrees of freedom.

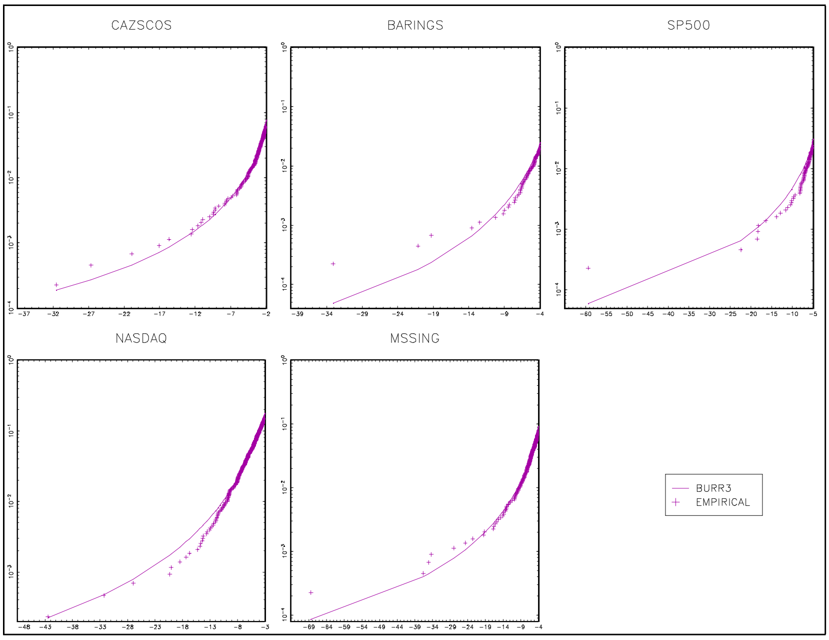

Figure 8: Left Tail of the CDF (in log-scale) for the annualized log-returns

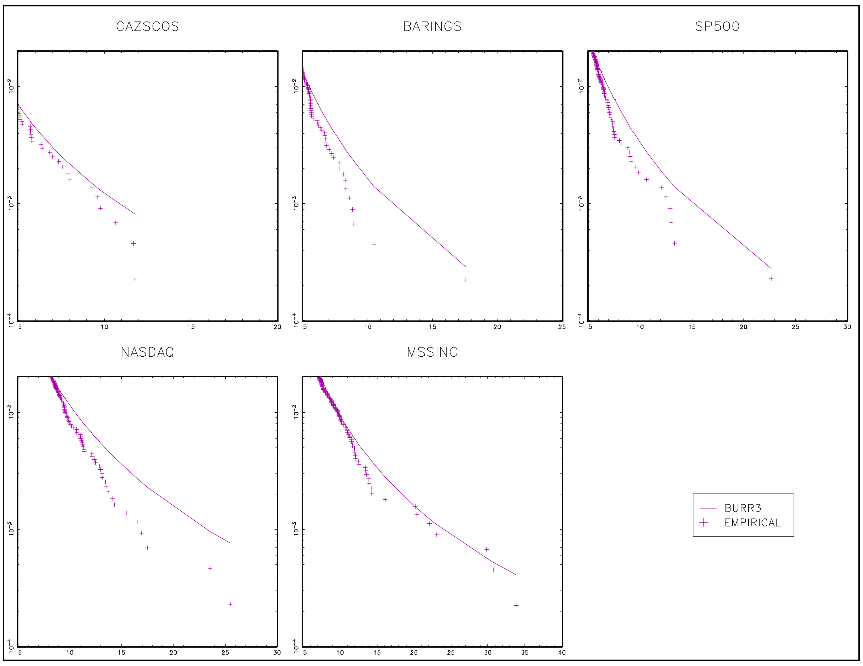

Figure 9: Right Tail of the survival CDF (in log-scale) for the annualized log-returns

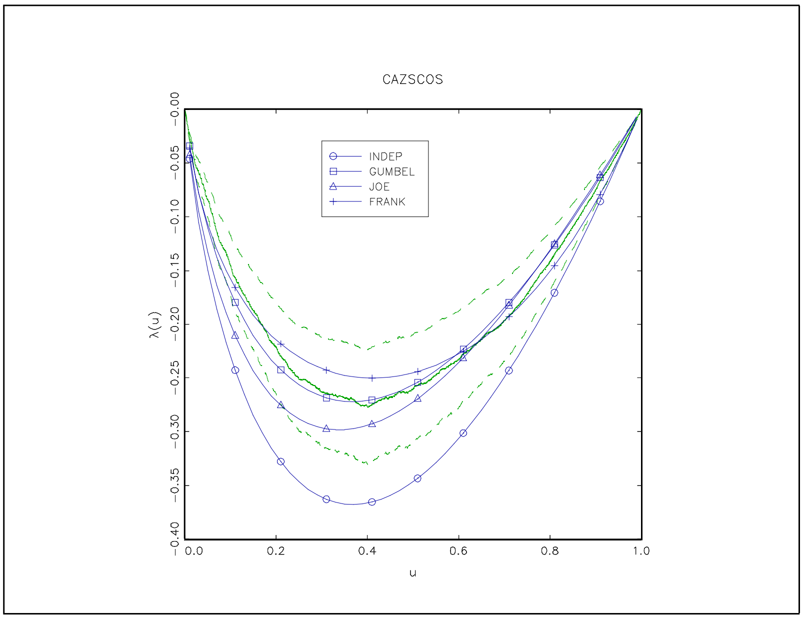

Figure 10: Empirical and fitted functions of $\lambda(u)$ for CAZSCOS. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

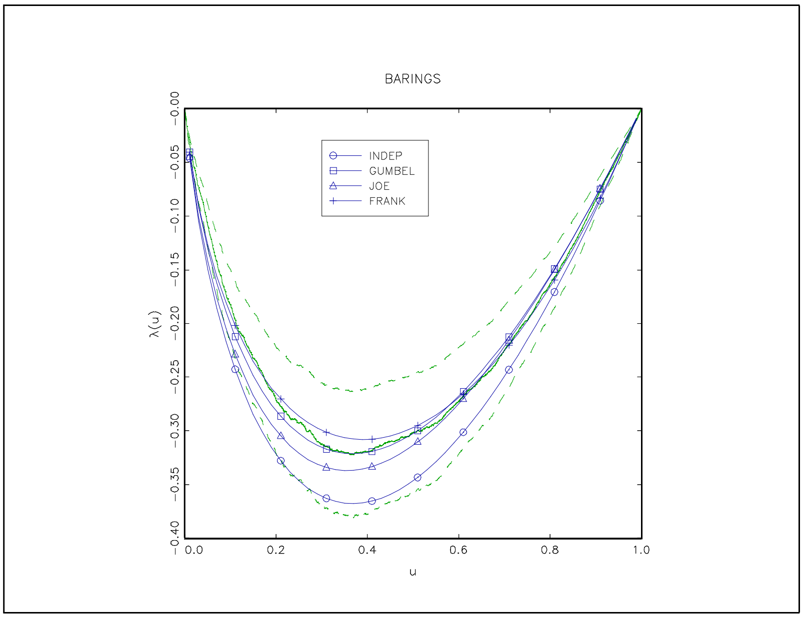

Figure 11: Empirical and fitted functions of $\lambda(u)$ for BARINGS. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

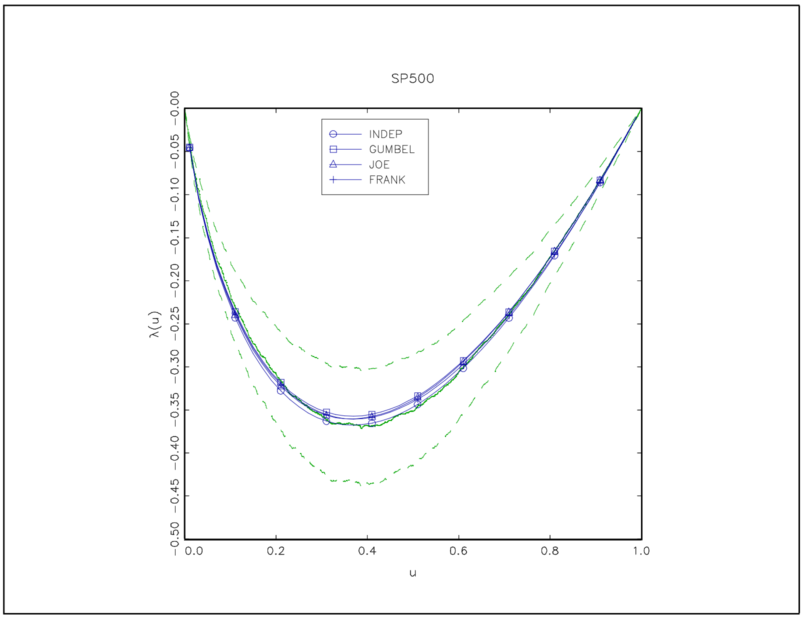

Figure 12: Empirical and fitted functions of $\lambda(u)$ for SP500. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

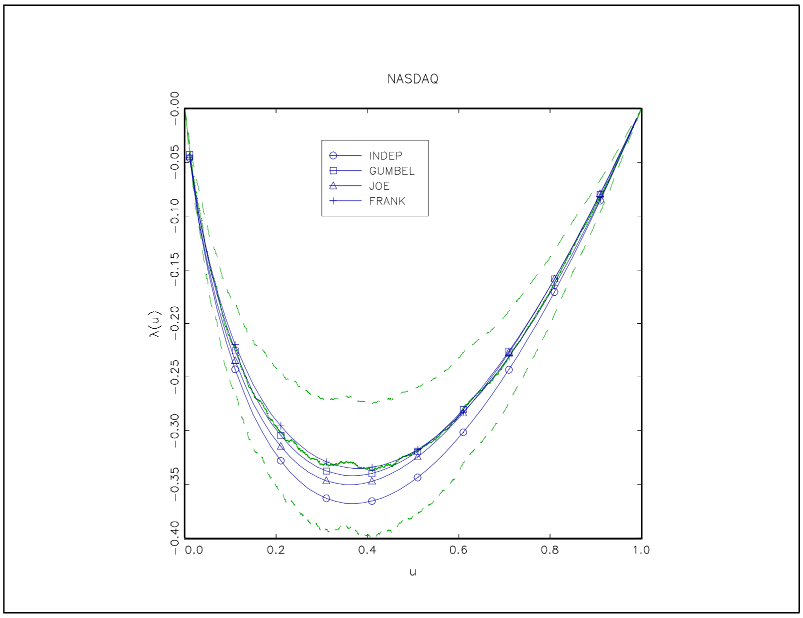

Figure 13: Empirical and fitted functions of $\lambda(u)$ for NASDAQ. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

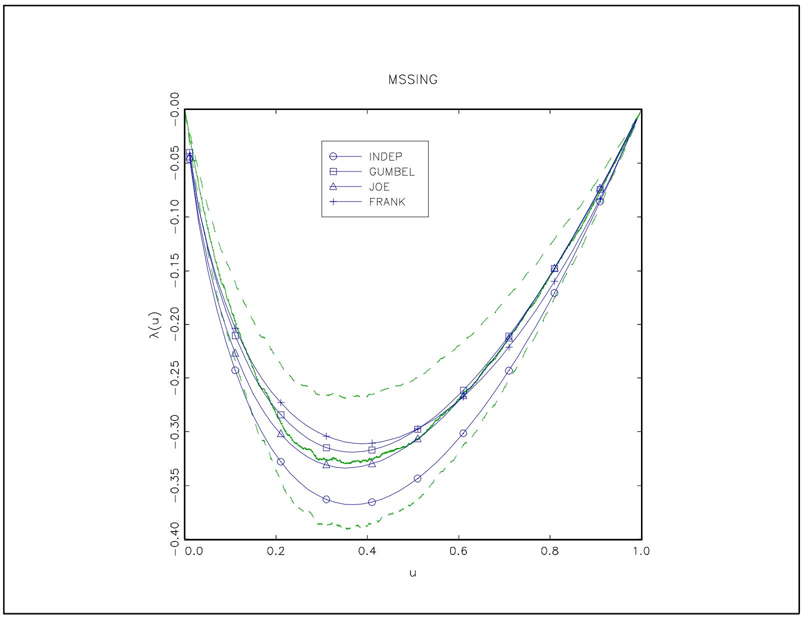

Figure 14: Empirical and fitted functions of $\lambda(u)$ for MSSING. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.