Measuring the Dependence between Non-Gaussian Financial Returns

Eric Bouyé and Mark Salmon

Financial Econometrics Research Centre
City University Business School
London

Empirical Finance is faced with two major problems:

- Returns are non-gaussian
- Need to consider joint risks or *multivariate* distributions

often making standard mean variance results very poor approximations to optimal results-
and this applies virtually everywhere

- Risk Management, Pricing, Hedging Portfolio construction,.....

Approaches to Constructing Multivariate Distributions

- Translation Method

Dates back to Edgeworth (1896); in general transform complicated pdf to normal-but can be used in reverse. Set of transformations, Johnson(*Biometrika*,1949), Johnson Family of Distributions; translation back to normality takes the form

$$z = \alpha + \beta J(x)$$

where z is standard normal; simplest case $z = \frac{x-\mu}{\sigma_x}, J(x) = x$,
extends to *bivariate* pdf where $z_i = \tilde{J}_i(x_i) = \alpha_i + \beta_i J(x_i)$

$$q(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{J}_1^2(x_1) - 2\rho\tilde{J}_1(x_1)\tilde{J}_2(x_2) + \tilde{J}_2^2(x_2)\right]\right\}$$

a range of different transformations lead to different systems of curves or families

of distributions
cf. Pearson, Fréchet, Charlier

Deriving Bivariate Distributions with Given Margins

Take marginal distribution for assets as given in deriving the bivariate distribution function.

Given marginals $F(x)$ and $G(y)$ then using the transformation

$$F(x) = \Phi(x_0) \text{ and } G(y) = \Phi(y_0)$$

where $\Phi(\cdot)$ is the standard normal distribution function then x_0 and y_0 are $N(0,1)$.

So we have the implied transformation given the marginals

$$x_0 = J_x(x) \equiv \Phi^{-1}(F(x))$$

$$y_0 = J_y(y) \equiv \Phi^{-1}(G(y))$$

Nataf(1962) showed any convenient distribution function could be used instead of multivariate normal. Mardia(1970) showed that the following pdf has nice properties;

$$\mathrm{h(x, y)} = \frac{f(x)g(y)}{\sqrt{1-\rho^2}} \exp\left\{-\frac{\rho}{2(1-\rho^2)}[\rho(J_x^2(x) + J_y^2(y)) - 2J_x(x)J_y(y)]\right\}$$

and provides sufficient conditions for h to be a pdf. Then becomes a question of fitting the parameters.

Notice that $|\rho(x_0, y_0)| \geq |\rho(x, y)|$

The Copula

- A copula is simply a function that links univariate marginals to their joint multivariate distribution or alternatively it is a joint distribution function with uniform marginals.

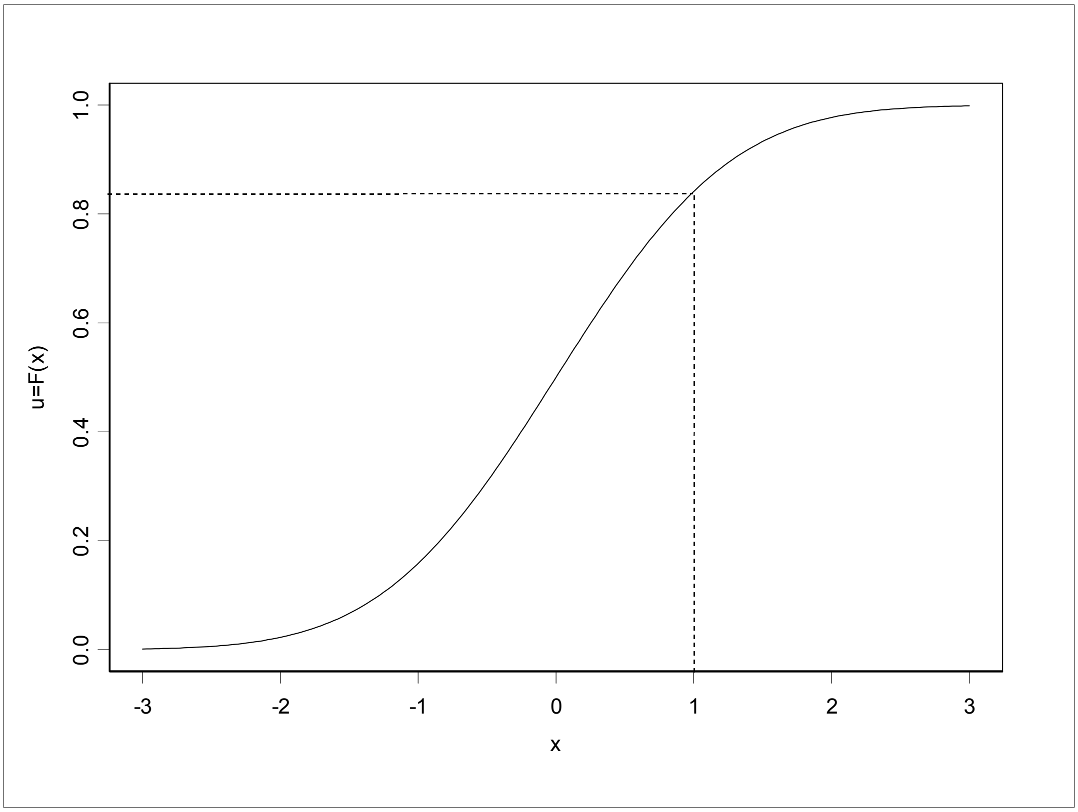
$$C(u_1, u_2, \ldots, u_N) = \Pr[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_N \leq u_N]$$

  with $U_1, U_2, \ldots, U_N$ being uniform random variables.
- Suppose we have a portfolio with N assets whose returns follow *arbitrary* univariate marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_N(x_N)$. The copula function C combines the marginals to give the joint density such that:

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1, x_2, \ldots, x_N)$$

  $F(x)$ is obviously a uniform random variable
- We define F^{-1} as the *pseudo-inverse* function *i.e.*

$$x = F^{-1}(u) \equiv \sup\{x | F(x) \leq u\}$$



- Major use of copula is in the construction of multivariate distributions– *modelling the joint distribution of different risks.*
- Notice that this elementary probability transform is simply the usual approach adopted for simulating data from an arbitrary distribution $F(x)$. In other words generate a random sample from a uniform$[0,1]$ distribution and then apply to each value the inverse function $F^{-1}(u) = x$. The resulting sample will be as if drawn from the distribution $F(x)$.
- Then we have simply that;

  $C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N))$

$$
\begin{aligned}
&= \Pr[U_1 \leq F_1(x_1), U_2 \leq F_2(x_2), \ldots, U_N \leq F_N(x_N)] \\
&= \Pr[F_1^{-1}(U_1) \leq x_1, F_2^{-1}(U_2) \leq x_2, \ldots, F_N^{-1}(U_N) \leq x_N] \\
&= \Pr[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_N \leq x_N] \\
&= F(x_1, x_2, \ldots, x_N)
\end{aligned}
$$

- Conversely, for a *given* multivariate distribution, there exists a copula function that links its marginals. Moreover, if the marginal distribution functions are continuous, the copula is *unique* (Sklar 1959).

  eg. Consider for $\delta > 0$, the distribution

  $$F(x,y) = \exp\left\{-\left[\exp(-x) + \exp(-y) - (\exp(\delta x) + \exp(\delta y))^{-\frac{1}{\delta}}\right]\right\}$$

  for $-\infty < x, y < \infty$. Let $y \to \infty$ and $x \to \infty$ in turn to generate its univariate marginals, $F_1(x) = \exp(-\exp(-x))$ and $F_1(y) = \exp(-\exp(-y))$. Then by substituting $u = F_1(x)$ and $v = F_2(y)$ we obtain the copula

  $$C(u,v) = uv\exp\left\{[(-\log u)^{-\delta} + (-\log v)^{-\delta}]^{-\frac{1}{\delta}}\right\}$$

- Since the multivariate distribution contains *all* the information that exists on *the dependence structure* between the variables the copula contains precisely the same information.
- Moreover since the copula is defined on the transformed *uniform* marginals it contains the information on dependence *irrespective* of the particular marginals of the underlying assets.
- Determine the marginal distribution of each asset- then estimate the copula from the data - contains all the information on both the type and degree of dependence between the assets to determine their *joint distribution* and hence assess *joint risks.*

The Literature

1. **Nelsen R.B.**,(1998) , *An introduction to Copulas*, Lecture Notes in Statistics 139, Springer Verlag.
2. **Joe H.**(1997), *Multivariate Models and Dependence Concepts*, Chapman and Hall, London.
3. **Embrechts, McNeil and Straumann**,*Correlation: Pitfalls and alternatives*
4. **Genst C. and J.MacKay**, (1986), *The Joy of Copulas: Bivariate distributions with Uniform Marginals,* American Statistician,40,280-283
5. **Frees E.W. and E.A. Valdez**, (1998), *Understanding Relationships using Copulas*,North American Actuarial Science Journal*, 2, 1-25.*

More Recently:

1. **Bouyé E., V.Durrleman, A.Nickeghbali, G.Riboulet and T.Roncalli**, (2000), *Copulas of Finance- A Reading Guide and some Applications,* FERC WP
2. **Embrechts P.,Lindskog F. and A.McNeil (2001) Modelling Dependence with Copula,**
   www.math.ethz.ch/finance
3. **Li D.X**.,(2000), *On Default Correlation: A Copula Function Approach,*Journal of Fixed Income
4. **Cherubini U. and E. Luciano**,(2000) *Value at Risk trade-off and capital allocation with Copulas* , http://web.econ.unito.it/gma/elisa.htm#wps
5. **Cherubini U. and E. Luciano**,(2000) *Multivariate Option Pricing with Copulas*, as above
6. **Frey R. and McNeil A**., (2001)*Modelling Dependent Defaults*, ETH Zurich
7. **Bouyé E., N.Gaussel and M. Salmon,** (2000)*Copulae and Time Dependence*, FERC WP.
8. **Sancetta A. and Satchell S**.,(2001) *Portfolio Construction and Optimisation using Bernstein Approximations to Copulae*, FERC Discussion Paper.
9. **Patton A,** (2000), *Modelling Time Varying Exchange Rate Dependence using Conditional Copula*, UCSD DP
10. **Rockinger M. and Jondeau E**., (2001), *Conditional Dependency of Financial Series: An Application of Copulas*, Mimeo HEC
11. Credit Lyonnais website
    http://gro.creditlyonnais.fr/content/rd/rd_math/copulas.htm#debut
    ...plus Bank of England, UBS Warburg,Credit Lyonnais, HSBC,Deutsche Bank,..........

Examples

Bivariate Gaussian

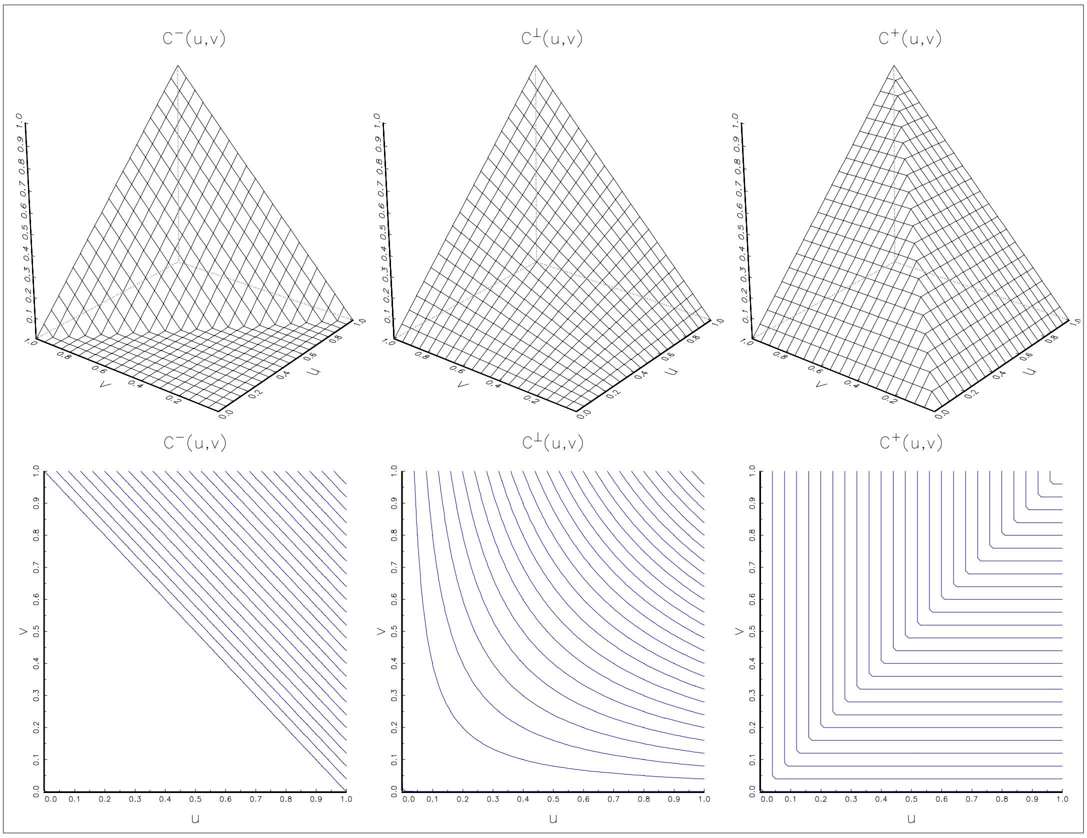
$$C(u, v; \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

with Φ_ρ the bivariate gaussian cdf and Φ^{-1} the inverse gaussian cdf.

$$c(x, y; \rho) = (1-\rho^2)^{-1/2}\exp\{-\frac{1}{2}(1-\rho^2)^{-1}[x^2+y^2-2\rho xy]\}.$$
$$\exp\{\frac{1}{2}[x^2+y^2]\}$$

with $x = \Phi^{-1}(u)$ and $y = \Phi^{-1}(v)$

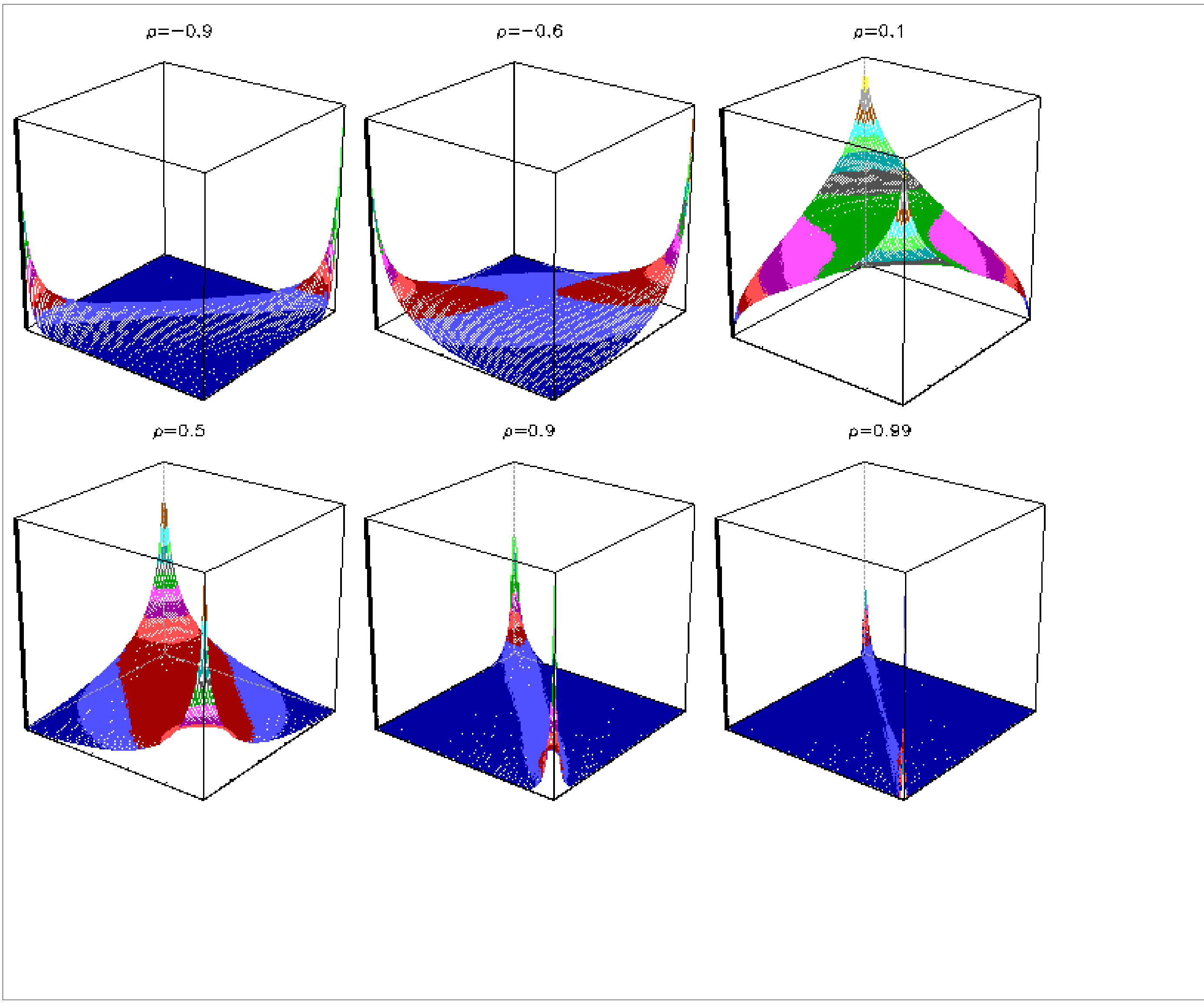
- Dependence measured by the *single* parameter ρ
- Multivariate Gaussian assumption used in mean variance portfolio theory, VaR,... amounts to assuming each asset follows a marginal Gaussian distribution *and* a *Gaussian* Copula.
- $C^- = C_{\rho=-1} < C_{\rho<0} < C_{\rho=0} = C^\perp < C_{\rho>0} < C_{\rho=1} = C^+$
  (C^-, C^+) are the Fréchet Bounds



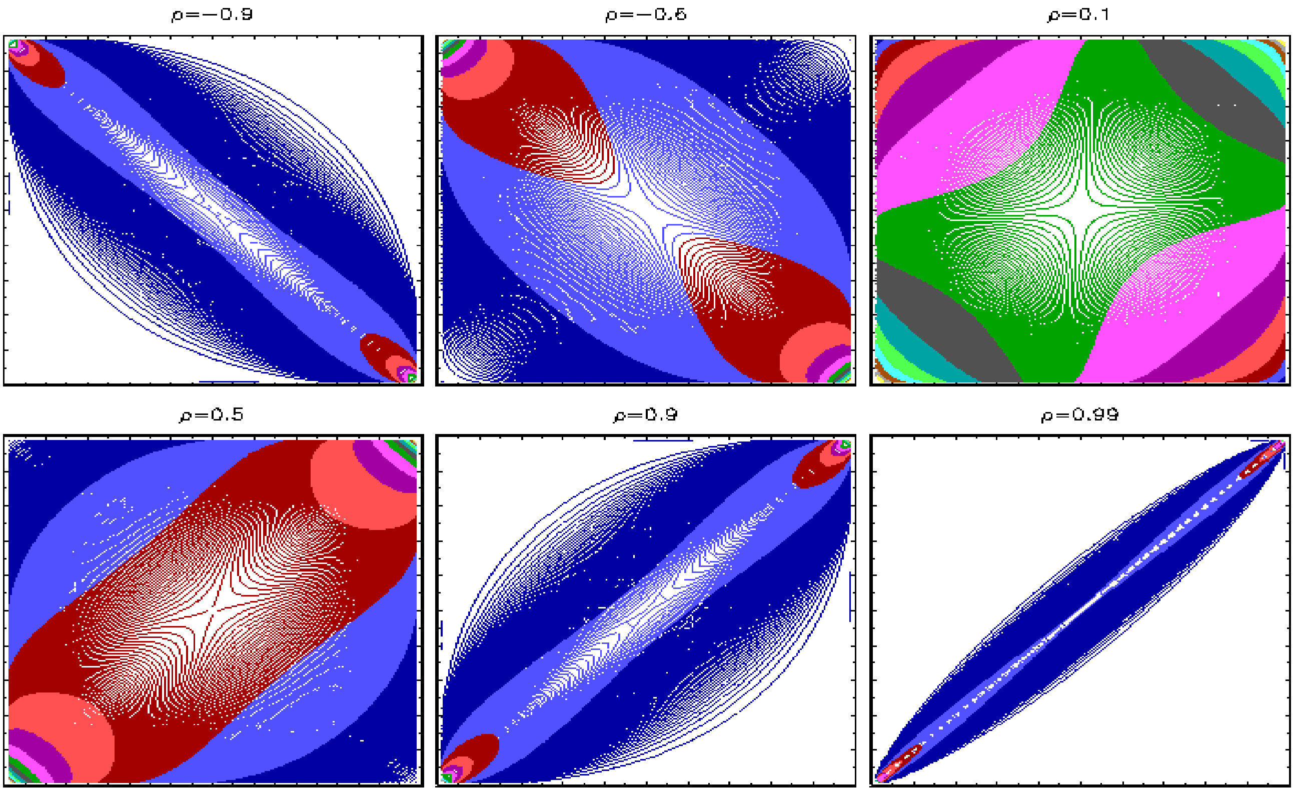
Frechet Bounds

$C(u, v) = uv \Rightarrow C(.2, .2) = 0.04$

Probability of both FTSE and SP500 being below 20^{th} percentile under independence copula.



Bivariate Gaussian Copula



Bivariate Gaussian Contours

Plackett(1965)

$$C(u,v;\delta) = \frac{1}{2(\delta-1)}\left(1+(\delta-1)(u+v)-[(1+(\delta-1)(u+v))^2-4\delta(1-\delta)uv]^{1/2}\right)$$

$$c(u,v;\delta) = \delta(1+(\delta-1)(u+v-2uv))[(1+(\delta-1)(u+v))^2-4\delta(1-\delta)uv]^{-3/2}$$

δ - the *dependency parameter*

Frank(1979)

$$C(u,v;\delta) = -\delta^{-1}\log\left(\frac{1-e^{-\delta}-(1-e^{-\delta u})(1-e^{-\delta v})}{1-e^{-\delta}}\right)$$

$$c(u,v;\delta) = \frac{\delta(1-e^{-\delta})e^{-\delta(u+v)}}{[1-e^{-\delta}-(1-e^{-\delta u})(1-e^{-\delta v})]^2}$$

$\delta > 0$ implies positive dependence, $\delta \to 0$ independence
$\delta < 0$ implies negative dependence, symmetric dependence

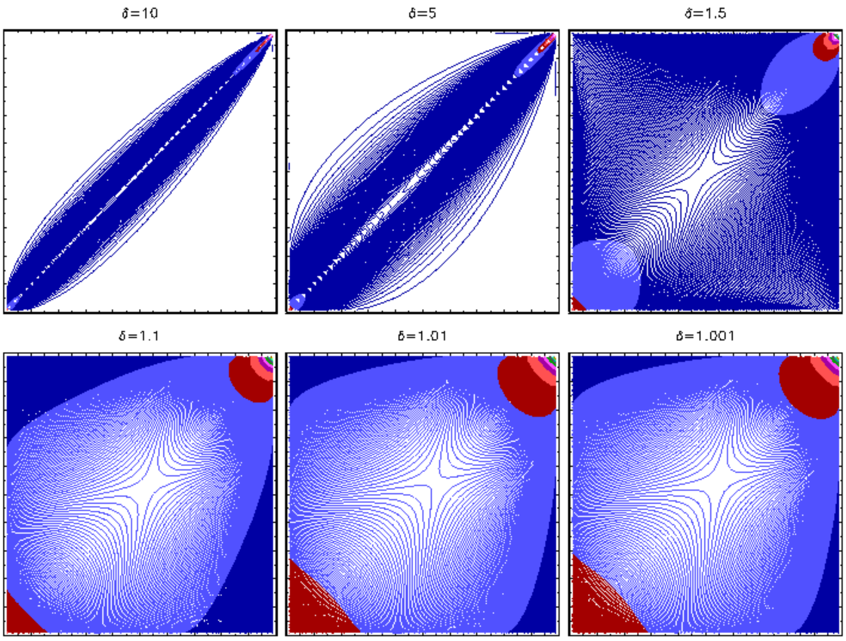
Gumbel(1960)

$$\textbf{CDF}: C(u,v;\delta) = \exp\{-((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta}\}$$

$$\textbf{PDF}: c(u,v;\delta) = C(u,v;\delta)(uv)^{-1}\frac{(\log u \log v)^{\delta-1}}{((-\log u)^{\delta} + (-\log v)^{\delta})^{2-1/\delta}} \times [((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta} + \delta - 1]$$

$$\textbf{CDFC}: C_u^{-1}(v;\delta) = u^{-1}\exp(-((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta})\left(1 + (\frac{\log v}{\log u})^{\delta}\right)^{1/\delta-1}$$

$\delta = 1$ implies independence and $\delta \to 0$ leads to perfect dependence
Increasing dependence at right tails
Gumbel-Hougard Copula $\delta \in (1,\infty)$



Gumbel-Hougard Copula

- A vast range of copula exist (see Joe)- many of which describe dependency with more than one parameter.

A two parameter example:

$$C(u,v;\theta,\delta) = \{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{\frac{1}{\delta}}\}^{-\frac{1}{\theta}}$$

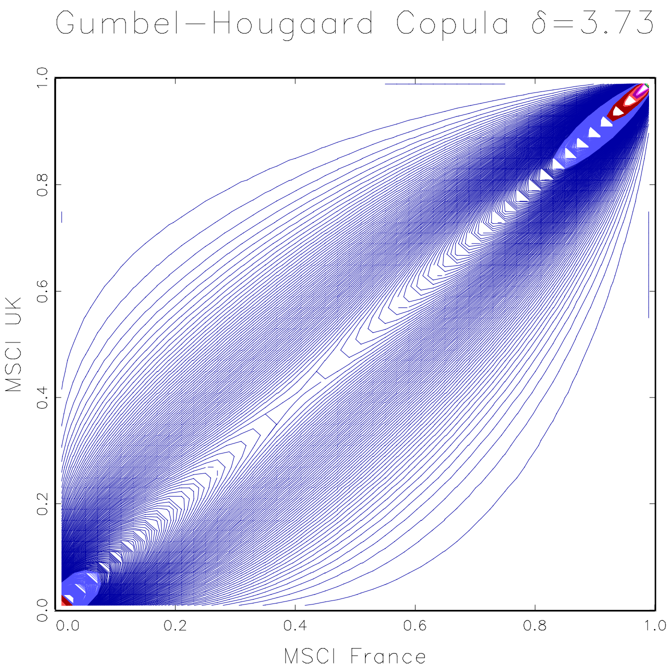
$$= \eta(\eta^{-1}(u) + \eta^{-1}(v))$$

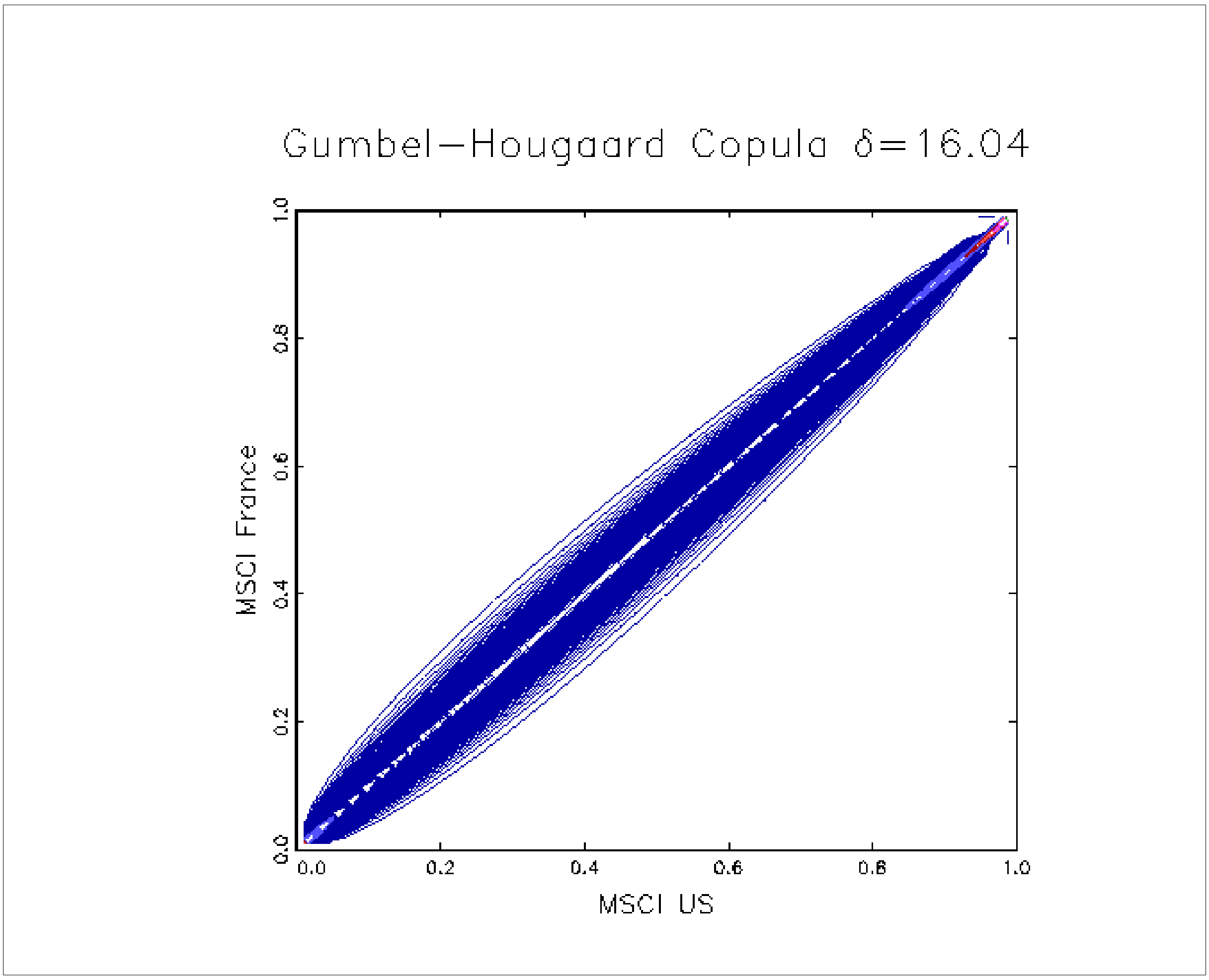
where $\eta(s) = \eta_{\theta,\delta}(s) = (1 + s^{\frac{1}{\delta}})^{-\frac{1}{\theta}}$

- LOWER TAIL DEPENDENCY PARAMETER = $2^{-\frac{1}{\delta\theta}}$
- UPPER TAIL DEPENDENCY PARAMETER = $2\text{-}2^{\frac{1}{\delta}}$ (Independent of θ)

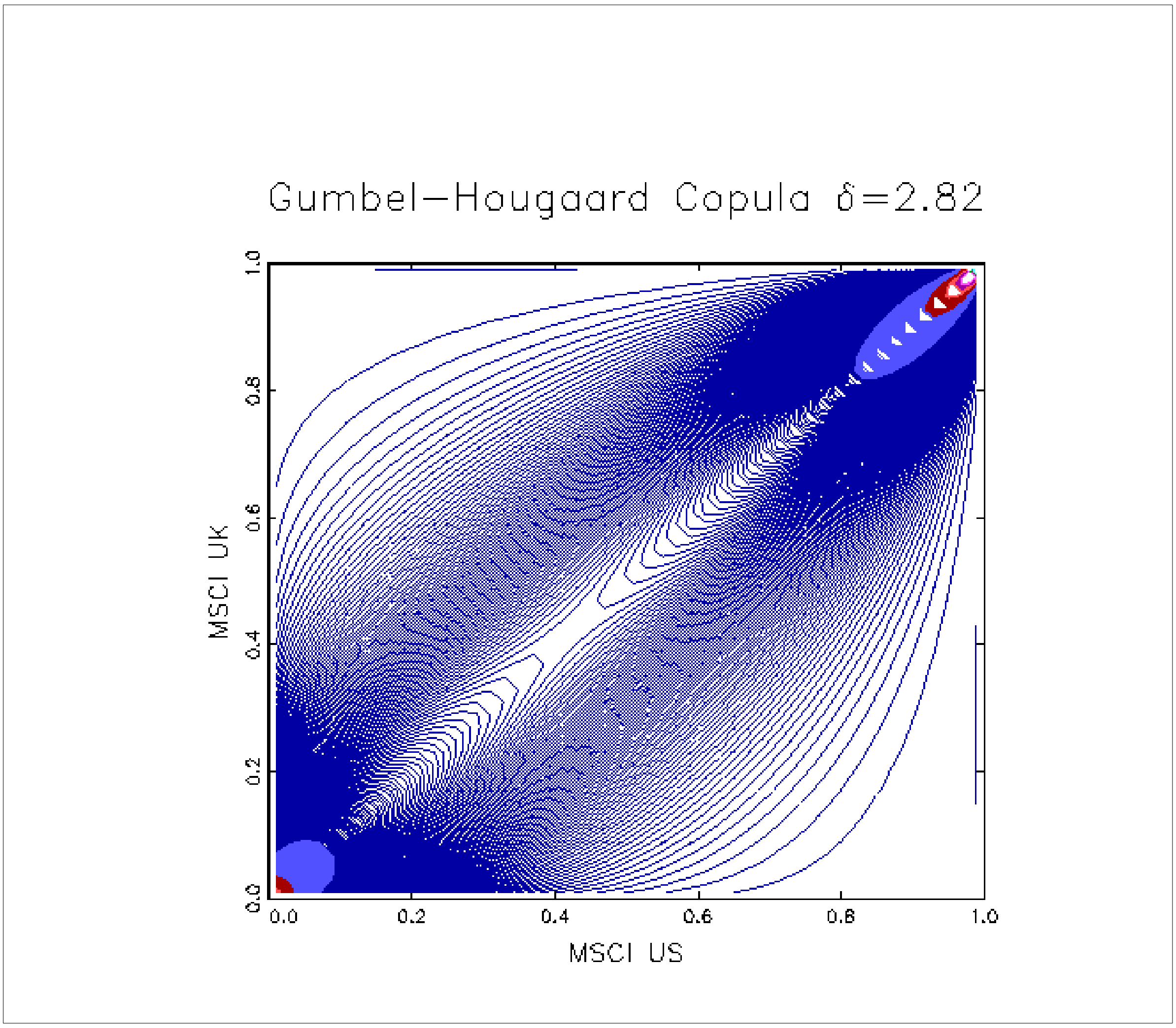
- CONCORDANCE INCREASES as θ INCREASES
- Joe suggests that multi-parameter copulae may be used "to capture more than one type of dependence"
- Dependency Measures can usually be expressed in terms of the parameters of the copula.

eg. one parameter for upper tail dependence and another for concordance– but the issue is more general– in general dependence measures will be functions of the copula's parameters– although they need not be *the* parameters of the copula.





Gumbel-Hougaard Copula δ=16.04
MSCI France
MSCI US
0.0
0.2
0.4
0.6
0.8
1.0



Measuring Dependency

Independence:

If the random variables $X_1, X_2, \ldots\ldots. X_n$ are independent then the copula function that links their marginals is the *product* copula

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1)F(x_2)\ldots, F(x_N) = C^{\perp}$$

- So *tests* for independence can be based on the distance of the empirical copula to the product copula.
- More generally the copula function is defined over the entire range of the random variables (transformed into the uniform$[0,1]$ space) and hence we *can examine the dependence structure through the entire range of the potential variation of the assets behaviour rather than through a single number such as the correlation*.– What are we interested in?

Criteria for Good Dependence Measures

Criteria that any measure of dependence δ between two continuous random variables X_1 and X_2 should satisfy include;

1. δ is defined for every pair (X_1, X_2),
2. $\delta(X_1, X_2) = \delta(X_2, X_1)$, symmetry
3. $-1 \leq \delta(X_1, X_2) \leq 1$,
4. $\delta(X_1, X_2) = 0$ if and only if X_1 and X_2 are independent,
5. $\delta(X_1, X_2) = 1$ if and only if each of X_1 and X_2 is almost surely a strictly monotone function of the other, COMONOTONIC (alternative forms exist ; for instance $\delta(X_1, X_2) = -1$ if and only if each of X_1 and X_2 is almost surely a strictly COUNTER-MONOTONIC function of each other).
6. $\delta(X_1, X_2) = \delta(T_1(X_1), T_2(X_2))$ with T_1 and T_2 almost surely strictly monotone functions,

The Inadequacy of Correlation

Pearson's Correlation Measure

$$\rho = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}$$

- Provides a measure of *linear* association
- $\sigma_x^2$and σ_y^2 have to be finite for ρ_{xy} to be defined; eg. extreme value type II (Fréchet) distribution with parameter $\tau = -\alpha^{-1}$ is such that $\int_0^\infty x^r \mathrm{d}F_X(x) = \infty$ for $r > \alpha$. So correlation is not defined in this quite reasonable and important case for financial applications
- Independence *always* implies zero correlation but the converse is only true for a multivariate gaussian (if another joint distribution for gaussian marginals is assumed, the converse does not hold).
- Weak correlations *do not imply* low dependence.

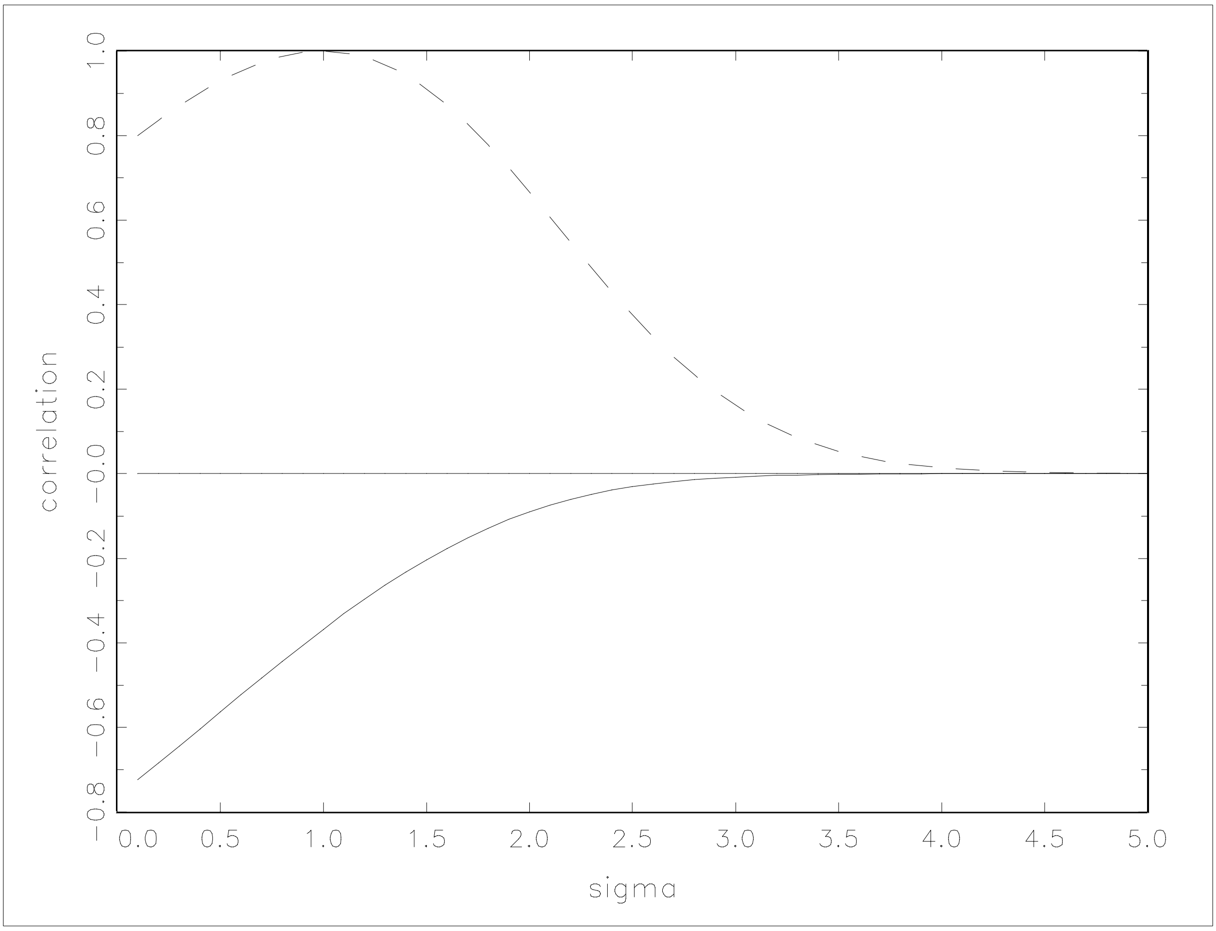
  (Embrechts *et al.* 1999, Example 5) consider two random variables X_1 ~Lognormal$(0, 1)$ and X_2 ~Lognormal$(0, \sigma^2)$. We can find that the minimum and maximum correlations between these two random varaibles are such that $\rho_{\min} = \rho(e^Z, e^{-\sigma Z})$ and $\rho_{\max} = \rho(e^Z, e^{\sigma Z})$ with $Z \sim \mathrm{N}(0, 1)$, it is possible to compute $\rho_{\min}$ and $\rho_{\max}$ explicitly as:

$$\rho_{\min} = \frac{e^{-\sigma} - 1}{((e-1)(e^{\sigma^2} - 1))^{1/2}}$$

  and

$$\rho_{\max} = \frac{e^{\sigma} - 1}{((e-1)(e^{\sigma^2} - 1))^{1/2}}$$

  These are shown in the Figure below and both min and max correlations tend to zero as sigma tends to infinity. So weak correlations do not imply low dependence.



- Correlation is not an *invariant* measure whereas the copula function *is* invariant. $\rho(X, Y) \neq \rho(\log X, \log Y)$

The fundamental reason why correlation fails as a general an invariant measure of dependency is due to the fact that *the Pearson Correlation coefficient depends not only on the copula but also on the marginal distributions*. Thus the measure is affected by changes of scale in the marginal variables.

$$\rho(X, Y) = \frac{1}{\sigma(X)\sigma(Y)} \iint_0^1 [C(u, v) - uv] \underbrace{dF}^{-1}(u) \underbrace{dG}^{-1}(v)$$

Alternative notions of Dependence

Many different forms of dependence between assets can exist that are simply not captured by correlation

co-skewness, co-kurtosis

for instance

- Patton (2001); Hu (2001) note that stock returns are more dependent during market downturns than during upturns- Sp500, DAX,Nikkei,Hang Seng much higher dependence in Bear Markets than in Bull markets; common sensitivity to bad news stronger than for good news.
- Relative Bear Market not volatility that describes dependence across markets
- asymmetric dependence given positive and negative returns.
- Longin and Solnik(2001); different dependencies for large and small movements in returns.

Measures of Concordance

Functions of the copula will be scale invariant under almost surely strictly increasing transformations⇒ invariant concordance measures

- Concordance between two random variables arises if large values of one variable arise with large values of the other and small values occur with small values of the other

Kendall's Tau

If (X_1, Y_1) and (X_2, Y_2) are independent and identically distributed random vectors with possibly different *joint* distribution functions H_1,and H_2 with copulae C_1 and C_2 respectively but with common margins. The population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance, Q

$$\tau = \tau_{X,Y} = [P(X_1 - X_2)(Y_1 - Y_2) > 0] - [P(X_1 - X_2)(Y_1 - Y_2) < 0]$$

Nelsen shows that this may be re-expressed in terms of the copulae as

$$Q = Q(C_1, C_2) = 4 \iint_{I^2} C_2(u,v) dC_1(u,v) - 1$$

Spearman's Rho

Let $\boldsymbol{R_i}$ be the rank of $\boldsymbol{x_i}$ among the $\boldsymbol{x}$**'s** and $\boldsymbol{S_i}$ be the rank of $\boldsymbol{y_i}$ among the $\boldsymbol{y}$**'s.** The Spearman rank order correlation coefficient is:

$$\rho_S = \frac{\sum_{i=1}^{n} (\boldsymbol{R_i} - \bar{\boldsymbol{R}})(\boldsymbol{S_i} - \bar{\boldsymbol{S}})}{(\sum_{i=1}^{n} (\boldsymbol{R_i} - \bar{\boldsymbol{R}})^2)^{1/2} (\sum_{i=1}^{n} (\boldsymbol{S_i} - \bar{\boldsymbol{S}})^2)}$$

which again my be expressed in terms of copulae as

$$\rho_C = \mathbf{12} \int\int_{[0,1]^2} (\boldsymbol{C(u,v) - uv}) \boldsymbol{dudv}$$

Spearman's rank correlation coefficient is essentially the ordinary correlation of $\rho(F_1(X_1), F_2(X_2))$ for two random variables $X_1 \sim F_1(.)$ and $X_2 \sim F_2(.)$.

- Essentially these two measures of concordance measure the degree of *monotonic* dependence as opposed to the Pearson Correlation which simply measures the degree of *linear* dependence
- They both achieve a value of unity for the bivariate Fréchet upper bound (one

variable is an increasing transformation of the other) and minus one for the Fréchet lower bound (one variable is strictly decreasing transform of the other). Functional dependence as opposed to linear dependence.

Gini index

$$\gamma_C = \mathbf{4}\left\{\int_0^1 C(u, 1-u)du - \int_0^1 (u - C(u,u))du\right\}$$

and Scheitzer and Wolff (Annals of Stats (1981)) have discussed three non-parametric measures based on L_1, L_2 and L_∞ distances

$$\sigma = 12 \iint_{I^2} |C(u,v) - uv| dudv$$

Hoeffding's measure:

$$\gamma_{X,Y} = \left(90 \iint_{I^2} [C(u,v) - uv]^2 dudv\right)^{\frac{1}{2}}$$

and

$$\kappa_{X,Y} = 4 \sup_{u,v \in [0,1]} |C(u,v) - uv|$$

There are formal relationships between these different measures (see Nelsen.) which can be used as the basis of nonparametric tests for the independence of two assets.

Daily log-returns for MSCI US, MSCI France and MSCI UK from 12/1987 to 12/1999.

Dependence Measures	Kendall	Spearman	Gini	Correlation
MSCI US - MSCI France	0.156	0.228	0.564	0.168
MSCI US - MSCI UK	0.180	0.264	0.604*	0.358
MSCI UK - MSCI France	0.397*	0.557*	0.470	0.527*

NB. Concordance may also be zero even if variables are dependent but bounded between 0 and 1 regardless of marginal distributions.

Gumble Copula indicated MSCI France and US were most highly associated in terms of δ for which high values suggests *tail* dependence important.

Positive Quadrant Dependent (PQD)

$$\Pr\{X > x, Y > y\} \geq \Pr\{X > x\}\Pr\{Y > y\}$$

So probability that two assets make large gains is greater than if they were independent or in terms of copula

$$C > C^{\perp}$$

Scaillet (2002) develops a test for PQD.

Survival Copulae

- Key role in credit risk management is the class of *survival copulae*. Assume two risks A and B with their respective survival times represented by two random variables T_A and T_B.
- Their survival functions are given by $S_A(t_A) = 1 - F_A(t_A) = \Pr\{T_A > t_A\}$ and $S_B(t_B) = 1 - F_B(t_B) = \Pr\{T_B > t_B\}$.
- Let C be the copula that links T_A and T_B, the joint density of the times to default of two risks. Then the joint survival function:

$$\begin{aligned} S(t_A, t_B) &= \Pr\{T_B > t_B, T_B > t_B\} \\ &= S_A(t_A) - S_B(t_B) - 1 + C(1 - S_A(t_A), 1 - S_B(t_B)) \end{aligned}$$

  Defining $\tilde{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ as the *Survival Copula* of T_B and T_A and we have

$$S(t_A, t_B) = \tilde{C}(S_A(t_A), S_B(t_B))$$

  $\tilde{C}$ "couples" the joint survival function to its univariate marginals and provides the means of addressing *the joint default risk*. Notice that there is an alternative notion of a joint survival function $\bar{C}$ for two uniform $(0,1)$ random variables whose joint distribution function is the copula C where

$$\begin{aligned} \bar{C}(u,v) &= \Pr[U > u, V > v] \\ &= 1 - u - v + C(u,v) \\ &= \tilde{C}(1-u, 1-v) \end{aligned}$$

Tail Area Dependence and Extremes

Environmental Science has developed an empirical dependence measure for extremes, λ - so called *tail dependence* where asymptotic independence is given by $\lambda = 0$ and $\lambda \in (0,1]$ for upper tail dependence.

λ_u is linked to the asymptotic behaviour of the copula:

$$\lambda_u = \lim_{\alpha \to 1^-} \Pr\{X_2 > VaR_\alpha(X_2) | X_1 > VaR_\alpha(X_1)\}$$
$$= \lim_{\alpha \to 1^-} \frac{1 - 2\alpha + \bar{C}(\alpha, \alpha)}{1 - \alpha}$$

or, alternatively (see Embrechts P., A. McNeil and D. Straumann (1999)):

$$\lambda_u = -\lim_{x \to 1^-} \frac{d(1 - 2x + \bar{C}(x,x))}{dx}$$
$$= \lim_{x \to 1^-} \Pr\{U_2 > x | U_1 = x\} + \lim_{x \to 1^-} \Pr\{U_1 > x | U_2 = x\}$$
$$= 2 \lim_{x \to 1^-} \Pr\{U_2 > x | U_1 = x\}$$

Applying the same transformation F_1^{-1} to both marginals, and $(X, Y)^\top \sim C(F_1(x), F_1(y))$,

$$\lambda_u = 2 \lim_{x \to \infty} \Pr\{F_1^{-1}(U_2) > x | F_1^{-1}(U_1) = x\}$$
$$= 2 \lim_{x \to \infty} \Pr\{Y > x | X = x\}$$

An alternative interpretation (given in Joe) : Let $\lambda(u)$ be viewed as a quantile dependent measure of dependence (Coles,Currie and Tawn (Lancaster University WP ,1999)) Then

$$\lambda_u(u) = \Pr[U_1 > u | U_2 > u] = \frac{\bar{C}(u,u)}{1 - u}$$

and

$$\lambda_u = \lim_{u \to 1} \frac{\bar{C}}{1 - u}$$

Where

$$\bar{C}(u_1, u_2) = \tilde{C}(1 - u_1, 1 - u_2)$$

NB.the tail area dependency measure λ_u depends on the copula and not on the marginal distributions.

- Quantile based measures of extreme dependence look highly promising tools for risk management.

Statistical Issues

Copulа estimation

Parametric

1. *Full Maximum Likelihood*: estimate parameters of copula and marginal distributions all at the same time.
2. *Inference Functions for Margins (IFM): Two step procedure in which you first estimate margins and then the copula parameters.*
3. *Conditional Maximum Likelihood (CML): Map marginal variables through their empirical distributions into uniform variables and then estiamte the copula parameters based on these input variables.*

Non-parametric

1. Deheuvels Empirical Copula
   ⇒No parametric prior on dependency pattern.

- A straightforward sample estimator for the copula: *the empirical copula* may be calculated as follows.
- Let $\{(x_{1k}, x_{2k})\}_{k=1}^{n}$ be a bivariate sample of size n , the empirical copula is defined as follows:

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_{\{x_{1k} \le x_{1(i)}, x_{2k} \le x_{2(j)}\}}$$

This estimator computes the frequency that both variables are greater than a specific pair of values (x_{1k}, x_{2k}).

- The *empirical copula frequency* is given by:

$$c_n\left(\frac{i}{n}, \frac{j}{n}\right) = \begin{cases} \frac{1}{n} & \text{if } (x_{1(i)}, x_{2(j)}) \text{ is an element of the sample} \\ 0 & \text{otherwise} \end{cases}$$

where C_n and c_n are related by

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \sum_{p=1}^{i}\sum_{q=1}^{j} c_n\left(\frac{p}{n}, \frac{q}{n}\right)$$

and conversely,

$$c_n\left(\frac{i}{n}, \frac{j}{n}\right) = C_n\left(\frac{i}{n}, \frac{j}{n}\right) + C_n\left(\frac{i-1}{n}, \frac{j-1}{n}\right) - C_n\left(\frac{i}{n}, \frac{j-1}{n}\right) - C_n\left(\frac{i-1}{n}, \frac{j}{n}\right)$$

and empirical copula estimates of Gini's gamma,g_n, Spearman's rho,r_n, and Kendall's tau , t_n, cf. (Nelsen 1998) concordance measures are given by

$$\begin{cases} g_n = \frac{2n}{[n^2/2]}\left(\sum_{i=1}^{n-1} C_n\left(\frac{i}{n}, 1-\frac{i}{n}\right) - \sum_{i=1}^{n}\left(\frac{i}{n} - C_n\left(\frac{i}{n}, \frac{i}{n}\right)\right)\right) \\ r_n = \frac{12}{n^2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_n\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n}\frac{j}{n}\right) \\ t_n = \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left(c_n\left(\frac{i}{n}, \frac{j}{n}\right)c_n\left(\frac{p}{n}, \frac{q}{n}\right) - c_n\left(\frac{i}{n}, \frac{q}{n}\right)c_n\left(\frac{p}{n}, \frac{j}{n}\right)\right) \end{cases}$$

2. Scaillet (2000): *Kernel based estimators* of copulae;

straightforward extension to smoothing empirical copula where real bounded functions $k_{ij}(x)$ defined on *R* such that for n variables estiamted at d points

$$\int k_{ij}(x)dx = 1 \quad i = 1,\ldots d\ ,j = 1,\ldots n$$

and

$$K_i(x,h) = \prod_{j=1}^{n} k_{ij}(x_j/h_j) \;\; i = 1,\ldots.d$$

where the bandwidth h is a diagonal matrix. The pdf of Y_{jt} at y_{ij} , ie. $f_j(y_{ij})$ is estimated by

$$\hat{f}_j(y_{ij}) = (Th_j)^{-1}\sum_{t=1}^{T} k_{ij}((y_{ij} - Y_{jt})/h_j)$$

and the estimates of the cumulative distribution of Y_{jt} at distinct points y_i is then given by

$$\hat{F}(y_i) = \int_{-\infty}^{y_{i1}} .. \int_{-\infty}^{y_{in}} \hat{f}(x)dx$$

Then to estimate the copula at distinct points $u_i, i = 1,\ldots,d$ use a simple plug in method, where

$$\hat{C}(u_i) = \hat{F}(\hat{\zeta}_i)$$

where $\hat{\zeta}_{ij} = \inf_{y\in R}\left\{y : \hat{F}_j(y) \geq u_{ij}\right\}$. The kernel estimate of the quantile of Y_{jt} at probability level u_j. Derives asymptotic properties under strong mixing conditions.

Approximations

Li, Mikusinski,Sherwood and Taylor (1997) present approximations of a copula that converge to the true copula.

Any copula can be approximated arbitrarily closely in the uniform sense by copulae that correspond to the deterministic dependence between a pair of random variables.

select spanning space of operators.

1. *Bernstein Polynomials*,
   Sancetta and Satchell(2001),
   Durrleman, Nikeghbali and Roncalli (2000)
2. *Checkerboard* and *Tent* approximations, Kulpa(1999)
3. *Partitions of Unity*, McKenzie (1994)

strong convergence but Bernstein and Checkerboard do not preserve upper tail dependence in convergence.

Choice between Copula:

- Different copula exhibit different dependency structures in different parts of the potential range of their margins, for instance
  - the *gaussian copula* implies when $\rho \neq 1$that the variables are asymptotically independent, ie. $\lambda_u = 0$ for $\rho < 1$
  - whereas the *t copula* implies extremes are asymptotically dependent for $\rho \neq -1$.
  - The *Frank copula* implies a *symmetric* dependence pattern, ie. the dependence is the same between positive returns as between negative returns.
  - *Clayton copula* implies higher dependence in bear markets.
  - *Gumbel copula* implies higher dependence in bull markets, increasing dependence in right tails so used to model extremes.
  - Asymmetric Dependence– becoming another clear stylised fact
- need careful empirical selection,
  - Standard approach: Goodness of Fit, AIC often used to select between parametric copula or non-parametric copula used.
  - Multivariate Encompassing Tests; Salmon (2002) Simulation based non-nested tests
- Consider using Mixtures of Copulae

$$C^M = \sum_{j=1}^{p} \alpha_j C_j(u_1, u_2, \ldots u_n)$$

such that $\sum_j \alpha = 1$ and C_j have different dependency structures.

Quantile Regression

- Need to move from Copula to *model building*
  - An important observation is that we can move from the multivariate distribution to derive all conditional quantile models;
    - standard regression conditional expectation
    - conditional quantile functions
    - conditional moment - ie ARCH

Without making any assumption on the form of the model

- entirely determined by structure of the Copula
  - so no need to make potentially spurious assumptions regarding linearity in the model etc.
  - General to Specific Modelling
- Since the copula captures the entire joint distribution it is not surprising that it can be used in considering the relationships between assets *away* from their conditional expectation– ie. away from the standard linear regression (around the mean).
- If we assume an Archimedean form of the copula ($C_\phi(u,v) = \phi^{-1}(\phi(u) + \phi(v))$) so that the conditional distribution of Y given $X_1.....X_k$ is given by

$$F_Y(y|x_1.....x_k) = \frac{\phi^{-k}\{c_k + \phi[F_y(y)]\}}{\phi^{-k}(c_k)}$$

  where $c_k = \phi[F_1(x_1)] + ... + \phi[F_k(x_k)] + \phi[F_Y(y)]$.
  and $\phi(\cdot)$ is some "generator" function.
- Elementary statistics show us that the regression function may be rewritten as

$$E(y|x_1.....x_k) = \int_0^\infty [1 - F_Y(y|x_1.....x_k)]dy + \int_{-\infty}^0 [F_Y(y|x_1.....x_k)]dy$$

- Genest(1987) shows that using Frank's Copula (with a generator function $\phi(t) = \ln(\frac{e^{\alpha t}-1}{e^{\alpha}-1})$) we can write the regression function directly as

$$E(X_2|X_1 = x) = \frac{(1 - e^{-\alpha})xe^{-\alpha x} + e^{-\alpha}(e^{-\alpha x} - 1)}{(e^{-\alpha x} - 1)(e^{-\alpha} - e^{-\alpha x})}$$

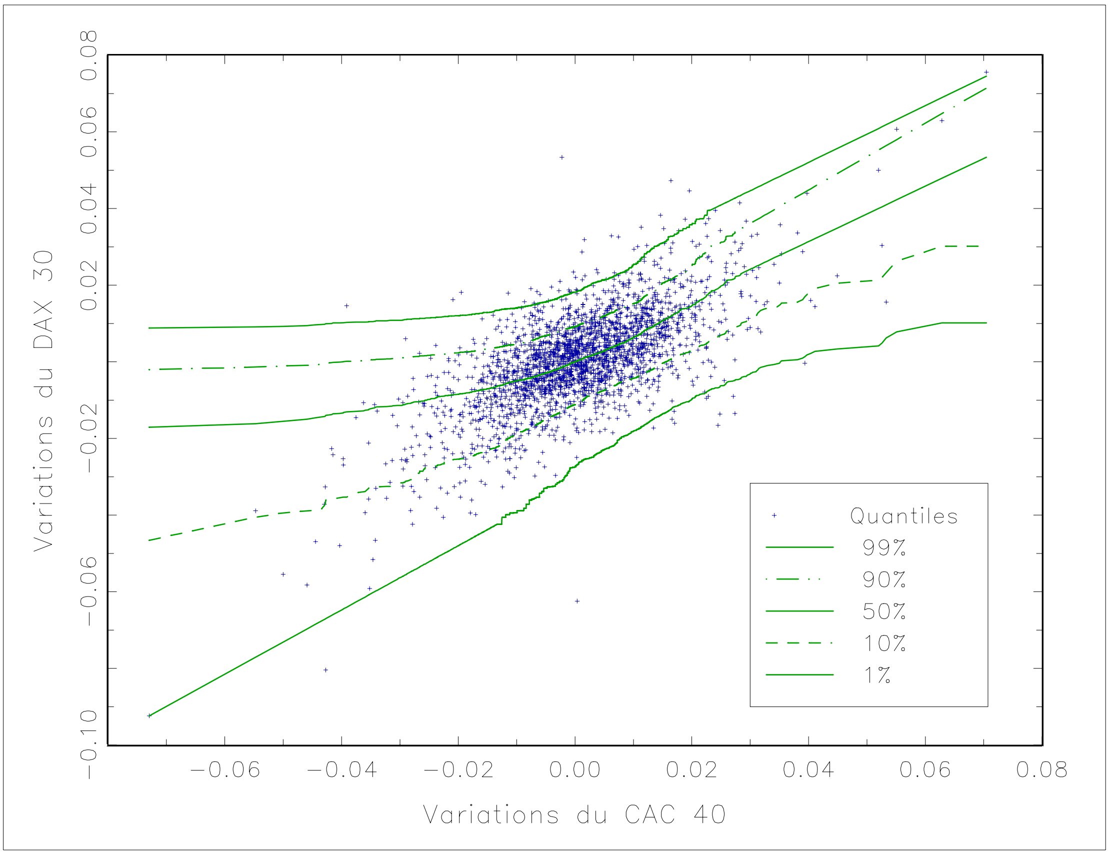
- Instead of calculating the conditional mean function we can compute the median or some other quantile of the conditional distribution. If we define the $p'th$ quantile to be the solution x_p of the equation

$$p = F_Y(y_p|x_1.....x_k)$$

  or for the bivariate case

$$p = F_Y(y_p|X_1 = x) = C_1[F(x), F_Y(y_p)]$$

  where C_1 is the partial derivative with respect to the first argument in the copula.
- So for a specified p and x value we can solve the equation above for the required quantile. The following plot shows the estimated relationships between the CAC40 and the DAX30 at various quantile levels



Variations du DAX 30
Variations du CAC 40
0.08
0.06
0.04
0.02
−0.02
−0.06
−0.10
−0.06
−0.04
−0.02
0.00
0.02
0.04
0.06
0.08
Quantiles
99%
90%
50%
10%
1%

Simulating the Joint Distribution of a number of assets

- Let H be a bivariate distribution function of two risks with F and G the marginal distribution functions and copula denoted by C. In other words, $H(x,y) = C(F(x), G(y))$.
- Sklar's theorem allows to use a simple method to generate bivariate returns (X,Y) by first simulating uniforms (U,V) whose joint distribution function is the copula C. We assume that F and G are strictly increasing.

STEP 1 Generate two independent uniforms (U,T)

STEP 2 Generate the two uniforms (U,V) by inverting the conditional distribution function such that $(u,v) = (u, C_u^{-1}(t))$ where:

$$C_u(v) = \Pr\{V \leq v \mid u = U\} = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u,v)}{\Delta u} = \frac{\partial C(u,v)}{\partial u}$$

STEP 3 The simulated returns are obtained by computing $(x,y) = (F^{-1}(u), G^{-1}(v))$

- This method might be extended to the multivariate case if there is a solution to the inversion of the conditional distribution functions. Let H be a N-variate distribution function and $(F_1, \ldots, F_N)$ the marginal distribution functions. Their copula is denoted C. In other words, $H(x_1, \ldots, x_N) = C(F_1(x_1), \ldots, F_N(x_N))$.

STEP 1 Generate N independent uniforms $(T_1, T_2, \ldots, T_N)$

STEP 2 Generate *recursively* the N uniforms $(U_1, U_2 \ldots, U_N)$ whose joint distribution function is the N-copula C as follows:

$$\begin{cases} u_1 &= t_1 \\ u_k &= C^{-1}_{(u_1,\ldots,u_{k-1})}(t_k) \qquad \text{for} \quad k = 2 \ldots N \end{cases}$$

where:

$$\begin{aligned} C^{-1}_{(u_1,\ldots,u_{k-1})}(u_k) &= \Pr\{U_k \leq u_k \mid U_1 = u_1, \ldots, U_{k-1} = u_{k-1}\} \\ &= \frac{\partial^{k-1} C(u_1,\ldots,u_k)}{\partial u_1 \ldots \partial u_{k-1}} \Big/ \frac{\partial^{k-1} C(u_1,\ldots,u_{k-1})}{\partial u_1 \ldots \partial u_{k-1}} \end{aligned}$$

STEP 3 The simulated returns are then obtained by computing $(x_1, \ldots, x_N) = (F_1^{-1}(u_1), \ldots, F_N^{-1}(u_N))$

Financial Applications

- *Li* at Riskmetrics is using Survival Copulae to measure *default dependency*; standard Riskmetrics approach implied the use of Gaussian Copula. Recently extended by *Frey and McNeil* (2001)
- *Hwang and Salmon* (2000) use copulae to capture the relationship between different *performance measures*, VaR and Tracking Error.
- *Cherubini and Luciano* (2001a,2002b) consider *option pricing* and *VaR*. Options based on multidimensional underlying. *Rosenberg* (1999)(2000) pricing multivariate contingent claims.
- *Bouyé and Salmon* (2000) are developing dynamic, nonlinear *quantile based risk models* using Copulae, cf. CAViaR (Engle and Manganelli, UCSD WP)
- Portfolio design, *Patton* (2001), *Sancetta and Satchell*(2001)
- *Kat and Salmon* (2002) looking at forms of dependency between hedge funds and market indices.
- *Aris Bikos* (2001),(Bank of England) uses copulae to construct *multivariate implied pdfs*- drawn from the option markets

  risk neutral copula.
- *Credit-Lyonnais*(1999-..) doing everything! pricing credit derivatives,VaR bounds...

Conclusions

- In many ways our life has just become much more *difficult*– We need to be absolutely clear about the form of dependency we are interested in measuring– around the mean, in the tails, PQD etc and a range of new dependency measures will be developed.
- Correlation can often tell very little about the relevant dependency pattern
- A huge range of potential applications are now possible, removing the multivariate Gaussian assumption– pricing etc approximations much better reflection of true underlying distributions and dependencies.
- Role of fat tailed *elliptic* distributions as opposed to Gaussian needs to be re-evaluated– retain mean variance analysis- implies *regime detection* and analysis of time varying skewness models of *Autoregressive Conditional Skewness* and *CaVaR.*
- A number of statistical issues need to resolved, in particular estimation of multivariate copula and the methods to discriminate formally between competing copulae.
- Critical importance of the general-specific approach to model building.