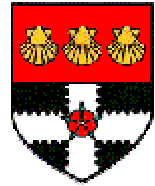




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Pricing Convertible Bonds By Simulation

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Abstract

Convertible bonds are complex hybrid securities subject to multiple sources of risk. Many exhibit exotic path dependent features. Monte Carlo simulation methods are usually the favorite choice for solving high-dimensional problems and pricing path dependent securities. This paper breaks away from the tradition established in the literature of pricing convertible bonds with finite difference and lattice methods, and suggests a simulation methodology for convertible bond pricing. We introduce the dividend process for convertible bonds, and formulate the pricing problem according to the probabilistic martingale approach. The proposed methodology deals with callable and puttable convertible bonds. The early exercise rules are estimated by means of least squares regressions as in Longstaff and Schwartz (2001). The accuracy of the simulation algorithm is tested in the context of a two factor model. The algorithm performs fairly well, and shows potential for further extension to include many complexities inherent in convertible bonds, as well as additional risk factors.

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1 Introduction

Convertible bonds are complex securities subject to equity risk, credit risk, and interest rates risk. Many exhibit features such as floating and step-up coupons, soft-call provisions, reset clauses and other exotic features described in Grimwood and Hodges (2002). Exchangeables and bonds convertible into the best of a basket of stocks are becoming more popular. Simulation methods are usually the favorite choice for solving high-dimensional problems and pricing path dependent derivatives. However, due to the fact that the optimal early exercise strategy is a free boundary problem, the literature on convertible bonds has only considered finite differences and lattice methods for solving the pricing equations. Given the restriction on the number of dimensions that can be handled by these numerical approaches, it has often been necessary to ignore certain risk factors while pricing convertible bonds. Furthermore, handling most path dependent features can become rather complex with these numerical procedures, and as a consequence, progress in pricing many important characteristics of convertibles was relatively slow. Recent advances in the literature on Monte Carlo valuation of American options has opened the door to developing fast and accurate simulation algorithms for pricing convertible bonds with path dependent features in a multi-factor setup.

This paper proposes the first, to our knowledge, Monte Carlo algorithm for pricing convertible bonds in the framework of intensity based credit risk models. The pathwise exercise decisions are estimated by means of regression methods as in Longstaff and Schwartz (2001).

The paper is organized as follows. After a literature review, we describe the financial securities market comprising a savings account, Treasury zero coupon bonds, corporate bonds, and common shares. While discussing credit risky instruments, we present an intensity based setup for credit risk modelling. In section 4, we introduce convertible bonds into our framework. Following the credit risk literature approach, we derive the dividend process for a plain vanilla convertible which leads to a risk-neutral valuation formula. Section 5 adapts the Least Squares Monte Carlo algorithm of Longstaff and Schwartz (2001) for convertible bonds valuation. The presence of put and call features requires careful handling of the issuer and investor's call, put and conversion decisions, which are discussed in detail. Then, in the last two sections, we apply the approach to price a range of convertible bonds in a context of a two factor model equivalent to Yigitbasioglu (2004). The results

obtained with the Monte Carlo approach are compared to prices computed with an accurate FD scheme. Our preliminary results provide evidence of viability of simulation pricing for convertible bonds. In particular, we show that the simulation pricing errors are fairly small for the considered scenarios.

2 Literature review

Given the many challenges associated with the valuation of convertibles and the importance of convertible debt market size¹, convertible bonds have been a subject of active academic research for the last three decades.

Most of the early research on convertible bonds pricing has adopted the firm value approach originated by Black and Scholes (1973) and Merton (1974). The key insight behind this approach is that all issued securities are derivatives on the firm value and can be priced using the contingent claim paradigm. Thus, the value of firm's assets is the main risk factor, and the default event corresponds to this value hitting a barrier, which is a function of the firm's liabilities. The firm value approach has been first applied to convertibles by Ingersoll (1977) who studied the optimal conversion strategy for investors and the optimal call policy for the issuer. The pricing Partial Differential Equation (PDE) was solved analytically for the case of non-callable and callable zero-coupon convertible bonds. It was shown that in the absence of dividends, a non-callable convertible bond is equivalent to a portfolio consisting of a straight bond and a European option².

Similarly, Brennan and Schwartz (1977) considered pricing a convertible bond in the firm value framework. A finite difference (FD) scheme was introduced to solve the pricing PDE for more general cases than in Ingersoll (1977). Subsequently, Brennan and Schwartz (1980) allowed for uncertainty in interest rates by assuming a Vasicek (1977) model for the short rate and considered more realistic firm capital structures than the early research. They also study the error caused by assuming constant interest rates, and provide a few examples where the magnitude of these errors does not justify the additional complexity introduced by modelling the short rate as a

¹As of the date of writing, the current CB market is worth more than US \$ 500bn. According to Morgan Stanley's ConvertBond.com database, there were 94 issues over \$ 500m by amount outstanding, and 380 issues between \$125m-\$500m by amount outstanding.

²The option's strike is equal to the face value plus the last coupon. The coupon was omitted in the original paper, which is equivalent to assuming that the investor can convert at maturity and still receive the coupon due at that time.

stochastic process.

Firm value models, whilst theoretically sound, are difficult to use in practice, mainly because they require the estimation of the unobservable firm value as well as its volatility. For this reason, McConnell and Schwartz (1986) introduced a single factor equity-based model. The authors point out that modelling the equity price, rather than the firm value, as a diffusion process precludes the possibility of bankruptcy. They insightfully proposed to use a default risk adjusted rate for discounting.

Most of the subsequent literature on convertibles has favored equity-based modelling, moving away from the firm value approach. Goldman Sachs (1994) used a Cox, Ross, Rubinstein (1979) stock price binomial tree where the discounting applied over a time-step is a conversion probability weighted average of the risky rate and the risk-free rate. The default event is not explicitly modelled but compensation for the credit risk is supposed to be included by the use of the so-called credit-adjusted discount rate.

In the same trend, Tsiveriotis and Fernandes (1998) proposed splitting the convertible into equity and bond components, each discounted at a different rate. This approach leads to a coupled single factor pricing PDE that was solved using less than ideal numerical methods. The model makes a hard to justify assumption that the equity part of the convertible bond is free from credit risk and thus should be discounted at the risk free rate.

Tsiveriotis and Fernandes model was extended by Yigitbasioglu (2001) to take into account interest rate and foreign exchange risk, allowing for calibration to the implied volatility term structure, both in equity and foreign exchange markets. The coupled PDE was solved using a stable and accurate finite difference scheme.

Recent advances in the credit risk literature, and especially the introduction of the so-called "reduced form" approach provides for more consistent handling of default risk in equity based models by allowing the stock price to jump downwards at the time of default. The reduced form models require to make explicit assumptions about the recovery rules applicable at the time of default.

The reduced form approach was pioneered by Jarrow and Turnbull (1995), and further extended by Jarrow, Lando and Turnbull (1997), Lando (1998), Madan and Unal (1998), Duffie and Singleton (1999) and many others. The approach is based on the assumption that the default time can be modelled as an exogenous and totally inaccessible stopping time. It was initially

viewed as ad-hoc, as opposed to the firm value based approach which models default as an endogenous event. It has only been recently that Duffie and Lando (2001) proved that a firm value model results in a reduced form model when imperfections in accounting information are taken into account. This study provides the reduced form modelling approach with a strong theoretical foundation.

In the context of reduced form models, Davis and Lischka (1999) proposed a two factor model with equity risk, and interest rate risk. The hazard rate was modelled as a deterministic function of the stock price.

Takahashi, Kobayashi and Nakagawa (2001) used the recovery of pre-default value assumption initiated by Duffie and Singleton (1999) to price a convertible bond with default risk. In particular, they use a formula from Duffie, Schroder and Skiadas (1996) according to which, subject to a certain 'no-jump' condition, the price of a defaultable security is equal to the risk-neutral expectation of discounted future cash-flows, where the discount rate has been adjusted to reflect the default risk.

Andersen and Buffum (2003) propose a single factor hazard rate model with proportional recovery of bond face value, and assume the hazard rate is proportional to the stock price. Their model is calibrated simultaneously to debt and option markets. Ayache, Forsyth, and Vetzal (2003) propose a single factor model that splits the convertible bond into equity and bond components, similar to Tsiveriotis and Fernandes (1998), but allows for the stock price to jump partially upon default, and the hazard rate to vary as a function of the stock price. They consider both, the market value and the face value recovery assumptions.

Grau, Forsyth and Vetzal (2003) price a convertible bond with call notice periods. In this case, the effective call price is in fact another CB with face value equalling the clean call price plus accrued interest, which must be solved separately³ at each time step in the FD scheme. Grau, Forsyth and Vetzal (2003) provide detailed pricing effects of the notice period using the model developed by Ayache, Forsyth, and Vetzal (2003).

Yigitbasioglu (2004) extends this framework to two factors, where the short rate dynamics follows a square root process as in Cox, Ingersoll and Ross (1985), and incorporates short-term and long-term uncertainty in the stock price volatility.

³Unless, of course, there are analytical prices for the "new" CB, i.e. when no dividends or put provisions exist in the notice period.

3 Financial securities market

Consider a frictionless economy with a trading horizon $[0, T^*]$. Our objective is to price a corporate convertible bond (CB) issued by a corporate entity ('XYZ') and maturing at time T , $T \leq T^*$. Assume that the underlying probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$ satisfies the "usual conditions" in the sense of Dellacherie and Meyer (1982). The filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ models the arrival of information over time.

In the following sections we describe the basic securities traded in the market. We assume that the market in non-defaultable securities comprises the savings account and Treasury zero coupon bonds of different maturities. The market for basic credit risky securities includes XYZ zero coupon bonds and XYZ common shares.

3.1 Non-defaultable instruments

3.1.1 Savings account

Assume that one of the traded securities in the market is the locally riskless savings account $\beta(t)$ defined by

$$\beta(t) = \exp \left(\int_0^t r_u du \right) \tag{1}$$

for some bounded and \mathbb{G} -progressively measurable short-term interest rate process r_t . Placing ourselves within the framework of an arbitrage-free financial model we postulate that the probability measure \mathbb{Q} is the equivalent martingale measure in the sense of Harrison and Kreps (1979). In particular, we say that the price process of any tradable security, which pays no coupons or dividends, follows a \mathbb{G} -martingale under \mathbb{Q} , when discounted by the savings account $\beta(t)$. The existence of an equivalent martingale measure is essentially equivalent to the absence of arbitrage in the market, as described by Harrison and Kreps(1979). As we do not require uniqueness of martingale measure at this stage, we shall make no assumptions here about market completeness.

3.1.2 Non-defaultable zero-coupon bonds

Let $B(t, T)$ denote the price at time t of a unit face value Treasury zero-coupon bond maturing at time T , $0 \leq t \leq T \leq T^*$. We define the price of a Treasury bond in a usual way as

$$B(t, T) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \middle| \mathcal{G}_t \right] \quad (2)$$

3.2 Credit-risky instruments

Unlike Treasury bonds, both corporate bonds and common shares are subject to the risk of default by the issuer. Therefore before we formally define the price process of XYZ bonds and stocks, we shall describe the way we model the default time.

3.2.1 Default time and default process modelling

We follow the reduced-form approach to default risk modelling and introduce the default time as a totally inaccessible \mathbb{G} -stopping time τ . $\{\tau > T\}$ is the event of no default before time T and assume that $\mathbb{Q}\{\tau < +\infty\} = 1$, $\mathbb{Q}\{\tau = 0\} = 0$ and $\mathbb{Q}\{\tau > t\} > 0$ for every positive time t . We introduce a jump process H associated with the default time by setting:

$$H_t = \mathbf{1}_{\{\tau \leq t\}}, \quad t \in \mathbb{R}^+. \quad (3)$$

The stochastic process H_t is equal to one if default has occurred and zero otherwise. We shall refer to H as default process or default indicator function. It is also assumed that there exists a strictly positive \mathbb{G} -predictable process λ such that the process M given by

$$M_t = H_t - \int_0^t \mathbf{1}_{\{\tau > s\}} \lambda_s ds = H_t - \int_0^{t \wedge \tau} \lambda_s ds \quad (4)$$

is a uniformly integrable \mathbb{G} -martingale under \mathbb{Q} . The process λ is referred to as the \mathbb{G} -marginal intensity of the stopping time τ under \mathbb{Q} or default intensity. We assume that the process λ satisfies the integrability condition

$$\mathbb{E}_{\mathbb{Q}} \left[\exp \left(\int_0^{\nu} \lambda_s ds \right) \right] < \infty \quad (5)$$

for some strictly positive non-random time ν .

3.2.2 Defaultable zero-coupon bonds

Let $P(t, T)$ denote the price at time t of a defaultable unit face value XYZ zero-coupon maturing at time T . We define $P(t, T)$ by the formula

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[\frac{\beta_t}{\beta_{\tau}} R_{\tau} \mathbf{1}_{\{\tau \leq T\}} + \frac{\beta_t}{\beta_T} \mathbf{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] \quad (6)$$

where R_{τ} denotes the cash flow received by the investor at the time of default. No specific assumptions about the size of recovery cash flow are made at this stage⁴.

3.2.3 Stock price process

Let $\tilde{S}(t)$ denotes the process followed by the XYZ stock price at time t prior to default time. At default time, assume the stock immediately loses a fraction $(1 - \delta)$ of its pre-default value. It is common in the literature to postulate that the recovery fraction δ for shares is zero (see Jarrow and Turnbull (1995)). It appears however that this assumption is not always supported by the empirical evidence. Beneish and Press (1995) report that, on average, stock prices drop by significantly less than 100 % upon a bankruptcy announcement. We define the XYZ stock price process as follows:

$$S(t) = \mathbf{1}_{\{\tau > t\}} \tilde{S}(t) + \mathbf{1}_{\{\tau = t\}} \delta \tilde{S}(\tau) \quad (7)$$

where the process $\tilde{S}(t)$ follows some stochastic process.

⁴For our purposes, specifying the corporate bond recovery rate becomes important only when it comes to calibrating the default intensity process to the term structure of XYZ outstanding debt.

4 Convertible bonds valuation

Assume that at time t , the company has an outstanding convertible bond maturing at time T , $t < T \leq T^*$. The CB promises to make discrete coupon payments at times T_1, T_2, \dots, T_n , and to repay the principal at maturity, such that $t < T_1 < T_2 < \dots < T_n = T$. Furthermore, at any time⁵ η , $t \leq \eta \leq T$, the investor may choose to convert the bond⁶ into a number \mathbf{a} of XYZ common shares or put it at the put price $K^p(\eta)$ if the CB is puttable at that time. The number \mathbf{a} is referred to as the conversion ratio. In the case of conversion, the investor gives up the future stream of payments promised on the convertible bond as well as the accrued interest.⁷ On the other hand, at any time⁸ ϕ , $t \leq \phi \leq T$, the issuer may choose to call the CB at the call price $K^c(\phi)$ if the CB is callable at that time.

Here, our motivation for introducing the exercise prices as functions of time is twofold: first, this allows us to take into account time-varying exercise prices, but it is also convenient for the notation in the subsequent presentation. For example, at time s if the CB is not puttable but callable, $K^p(s) = 0$ and $K^c(s) = X$, where X is the effective call price at time s . We also generally refer to effective exercise prices as opposed to “nominal” exercise prices. The “nominal” exercise prices are the ones written in the CB contract, while effective call prices are the relevant values from a pricing perspective. For example, at a given time, the effective call price will be the “nominal” call price plus any accrued interest. Also in the case of a call notice period, the effective call price would be the price of a non-callable CB with maturity the end of the notice period, as observed by Grau, Forsyth and Vetzal (2003).

A convertible bond is a defaultable security. It is assumed that if the default occurs before the CB maturity, conversion or put time, and the call time, the investor is entitled to collect a fraction δ^* of the CB face value⁹ or to convert the bond into shares thus collecting their recovery value, all future promised payments being canceled. Denoting $\pi = \eta \wedge \phi$ and

⁵From now on, η will be used to refer to the conversion or put date.

⁶Here we discuss a general case in which conversion is allowed at any time, i.e. conversion option is of American type. There are however convertible bonds with a conversion option of Bermudan type and bonds that can only be converted after a certain future date.

⁷In particular, in the case of conversion at maturity the investor does not have the right to collect the coupon due at that time.

⁸From now on ϕ will be used to refer to the call date.

⁹As observed in Andersen and Buffum (2002), the assumption of recovery of a fraction of Face value seems to be more consistent with typical bankruptcy proceedings.

$ex(\pi) = \begin{cases} c, & \text{if } \pi = \phi \\ p, & \text{otherwise} \end{cases}$ the dividend process D of a convertible bond can now be defined as

$$\begin{aligned} D_t &= \int_{]0,t]} (1 - H_u) \mathbf{1}_{\{u < \pi\}} dA(u) \\ &+ \int_{]0,t]} \max(\mathbf{a}\delta\tilde{S}(u), F\delta^*) \mathbf{1}_{\{u \leq \pi\}} dH_u \\ &+ \max(\mathbf{a}\tilde{S}(t), K^{ex(\pi)}(t)) (1 - H_t) \mathbf{1}_{\{t = \pi\}} \end{aligned} \quad (8)$$

where F is the face value of the bond and the process A denotes the stream of promised cash flows, i.e. coupon and principal payments. Assuming the promised payments are made discretely, A is a jump process. All promised payments are stopped after default, call, put, or conversion. The middle term in the sum represents the payment upon default, which is the maximum of the fraction δ^* of bond's face value or the conversion ratio multiplied by the share value immediately after default. The last term describes the payment at call, put or conversion time.

4.1 Risk-neutral pricing

In a complete market model, the CB's price being unique, we can apply the risk-neutral valuation formula to define the ex-dividend price $CB(t, T)$ of the convertible bond at time t , prior to conversion, call, or put date as

$$\begin{aligned} CB(t, T) &= \sup_{\eta \in \Psi} \inf_{\phi \in \Xi} \mathbb{E}_{\mathbb{Q}} \left[\int_{]t, T]} \frac{\beta_t}{\beta_u} dD_u \middle| \mathcal{G}_t \right] \\ &= \inf_{\phi \in \Xi} \sup_{\eta \in \Psi} \mathbb{E}_{\mathbb{Q}} \left[\int_{]t, T]} \frac{\beta_t}{\beta_u} dD_u \middle| \mathcal{G}_t \right] \end{aligned} \quad (9)$$

where Ψ denotes the set of all feasible conversion or put strategies, and Ξ denotes the set of all feasible call strategies, which are stopping times taking values in $[t, T]$. It follows from the definitions above that the ex-dividend price of the CB is zero at time t if all dividend payments after time t are zero.

5 Monte-Carlo algorithm for CB pricing

The Monte Carlo pricing algorithm for convertible bond is based on the least-squares approach developed by Longstaff and Schwartz (2001). The objective of the algorithm is to provide a pathwise approximation to the exercise rules for all options embedded in the convertible bond. For example, in the case of a vanilla CB, at every conversion time before default the holder of the CB compares the payoff from immediate conversion to the expected present value of future payoffs from the bond and converts if the immediate payoff is higher. Thus the optimal conversion rule is essentially determined by the conditional expectation of discounted future payoffs from the bond. The key insight of Longstaff and Schwartz (2001) is that the conditional expectation can be estimated using the information contained in a sample of simulated paths by means of a simple regression. More specifically, the conditional expectation function is assumed to belong to the \mathbb{L}^2 -space of square-integrable functions relative to some measure. Since \mathbb{L}^2 is a Hilbert space, it has a countable orthonormal basis and thus the conditional expectation function can be approximated as a linear function of the first $N < \infty$ elements in the basis. The basis functions are chosen to be functions of values of the state variables. The ex post realized payoffs from continuation are then regressed on the first N elements of the basis in order to determine the coefficients in the linear function approximating the conditional expectation¹⁰. We next describe the application of the Longstaff-Schwartz approach to pricing defaultable convertible bonds.

5.1 A backward recursion algorithm

The pricing equation can be discretized naturally by considering a set of finite number of stopping times $\bar{\mathcal{T}} = \{t, t_1, \dots, t_K = T\}$. This leads to the following approximation:

$$\overline{CB}^K(t, T) = \sup_{\eta \in \bar{\mathcal{T}}} \inf_{\phi \in \bar{\mathcal{T}}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=1}^K \frac{\beta_t}{\beta_{t_j}} (D_{t_j} - D_{t_{j-1}}) \middle| \mathcal{G}_t \right] \quad (10)$$

It can be shown from dynamic programming arguments¹¹ that $\overline{CB}^K(t, T)$

¹⁰Details of the convergence of the Longstaff and Schwartz algorithm can be found in the original paper and will not be addressed here.

¹¹We do not present a proof here, but this result is quite intuitive and should make

satisfies the following recursion:

$$\overline{CB}^K(T, T) = \max\left(\mathbf{a}\tilde{S}(T), F + \kappa\right) \mathbf{1}_{\{\tau > T\}} + \max\left(a\delta\tilde{S}(T), \delta^*F\right) \mathbf{1}_{\{\tau = T\}} \quad (11)$$

$$\begin{aligned} \overline{CB}^K(t_i, T) &= \max\left\{\min\left(\mathbb{E}_{\mathbb{Q}}\left[\frac{\beta_{t_i}}{\beta_{t_{i+1}}}\overline{CB}^K(t_{i+1}, T)|\mathcal{G}_{t_i}\right], K^c(t_i)\right), \mathbf{a}\tilde{S}(t_i), K^p(t_i)\right\} \mathbf{1}_{\{\tau > t_i\}} \\ &+ \max\left(a\delta\tilde{S}(t_i), \delta^*F\right) \mathbf{1}_{\{\tau = t_i\}} \text{ for } i < K \end{aligned} \quad (12)$$

Hence, ideally the set of discrete stopping times should include the default time, but assuming that the recovery payoff is received at the first date in the set following the default time should be enough for reasonably small time steps.

This backward recursion algorithm can be used to compute $\overline{CB}^K(t, T)$ where the cash flows can be computed using simulated paths. For this purpose, we approximate the conditional expectations involved by fine-tuning the Least Squares Monte Carlo approach to adapt it to the convertible bond problem.

5.2 Approximating the conditional expectations

Following the Longstaff-Schwartz methodology, we assume that at time t_{K-1} the unknown functional form of $\mathbb{E}_{\mathbb{Q}}\left[\frac{\beta_{t_i}}{\beta_{t_{i+1}}}\overline{CB}^K(t_{i+1}, T)|\mathcal{G}_{t_i}\right]$ can be represented as a linear combination of a countable set of \mathcal{G}_{t_i} -measurable basis functions:

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{\beta_{t_i}}{\beta_{t_{i+1}}}\overline{CB}^K(t_{i+1}, T)|\mathcal{G}_{t_i}\right] = \sum_{j=0}^{\infty} \alpha_j L_j(X_{t_i}) \quad (13)$$

where $L_j(\cdot)$ is the j^{th} basis function and X_{t_i} denotes the value at time t_i of the vector of simulated state variables. X may contain the following variables: the underlying stock price, the value of the short rate, the value of the default intensity, and/or any other state variable that is relevant to the problem at hand. In practice we use only the first $N < \infty$ basis functions to approximate the conditional expectation. We also note that the discounted ex-post realized cash flows can be written as:

sense by itself.

$$\frac{\beta_{t_i}}{\beta_{t_{i+1}}} \overline{CB}^K(t_{i+1}, T) = \mathbb{E}_{\mathbb{Q}} \left[\frac{\beta_{t_i}}{\beta_{t_{i+1}}} \overline{CB}^K(t_{i+1}, T) | \mathcal{G}_{t_i} \right] + \epsilon \approx \sum_{j=0}^N \alpha_j L_j(X_{t_i}) + \epsilon \quad (14)$$

Once the subset of basis functions has been specified we proceed to estimate the coefficients α_j in the equation above by standard Least Squares regression methods using the simulated paths of state variables. Specifically, we solve the following minimization problem:

$$\hat{\alpha} = \arg \min_{\alpha} ({}^t \epsilon \epsilon | \epsilon = \gamma - \mathbf{Y} \alpha) \quad (15)$$

where the vector γ contains the realized cash flows $\overline{CB}^K(t_{i+1}, T)$ for the simulated paths, the matrix \mathbf{Y} contains the values of N basis functions for these paths, and α is the vector of coefficients in the equation (14).

5.3 Implementation issues and proposed solutions

The accuracy of the backward recursion as presented above depends critically on the quality of the approximation of the conditional expectations. The simplest choice of basis functions is to take powers of the state variables and cross-products. This is normally good enough for pricing American put options, specially if one follows Longstaff and Schwartz suggestion of including only the in-the-money paths in the regression, as these constitute the region where we are interested in approximating the conditional expectation. In the case of the CB, one can easily notice that the conditional expectation as a function of the state variables will typically take on different shapes in different regions of the state variables domain. Indeed, for a certain range of values of the underlying stock price, the CB will look similar to a corporate bond, while for another range of values it will look more like a call option, etc. In this case, the estimated parameters from the above regression will typically lead to a poor approximation of the conditional expectation function by achieving a compromise between the different regions. Indeed conducted experiments for a model described below, have revealed that errors in excess of 1.5% are common for cases where the level of stock volatility is higher than 20%. An easy solution is to treat different regions

differently¹². This can be done by exploiting specific knowledge about the instrument by observing that in certain regions of the state variables space, not all the embedded options have to be considered.

For simplicity let's assume no dividend, then it is never optimal to convert unless the CB is called. Observe also, that in this case for values of the conversion value¹³ which are higher than the effective call price, there is no need to approximate the conditional expectation: the CB will simply be called, and the investor converts. When the conversion value is between the call price and the put price, it is also never optimal to put the CB, but we need to run a regression to approximate the conditional expectation function so as to determine for which paths the CB would be called. Finally for the paths for which the conversion value is lower than the put price, we need to run a different regression to determine which paths would optimally lead to the exercise of the put option. Thus in the simple case of no-dividend paying stock, we have three sub-regions, but only two require approximating the conditional expectations function to determine the embedded options' exercise rules.

In the case of continuous dividends, the same three regions hold, but now the conversion option needs to be taken into account. The same rules hold when the conversion value is higher than the effective call price or lower than the effective put price. The only difference in this case is in the region where the conversion value is between the two exercise prices. Indeed, in the presence of dividends, opportunities for exercising the conversion option might also arise. Therefore, we are still in the presence of three sub-regions with only two that require regressions at every discretisation point, but we need to check for conversion opportunities in the sub-region between the exercise prices. In the next section, we specify the model used to test the accuracy of this improved simulation algorithm.

6 A two factor model for accuracy tests

As a test bed for our methodology we use a two factor model with stochastic stock price and short interest rate. We begin by specifying the assumptions made about the dynamics of the stock price prior to default, $\tilde{S}(t)$, the

¹²Possible approaches include using piece-wise polynomials to approximate the conditional expectations or semi-parametric approaches (B-splines).

¹³The conversion value simply refers to: $\mathbf{a}S(t)$

dynamics of the short rate r , and the process for the \mathbb{Q} -default intensity λ .

6.1 State variables' dynamics

Assume that, prior to default, the stock price follows a geometric Brownian motion under \mathbb{Q} :

$$\tilde{S}(t) = S(0) \exp \left(\int_0^t \mu_u du + \sigma W_t \right) \quad (16)$$

W^1 is a \mathbb{G} -Brownian motion under \mathbb{Q} and the drift μ is given by

$$\mu_t = \left(r_t - q + (1 - \delta)\lambda - \frac{1}{2}\sigma^2 \right)$$

with q denoting a continuous dividend yield and the \mathbb{Q} -default intensity λ is assumed to be constant.

Assume that the short rate follows a square root process, as in Cox, Ingersoll and Ross (1985):

$$\begin{aligned} dr_t &= \vartheta(\theta - r_t)dt + \nu\sqrt{r_t}dW_t^2 \\ r_0 &> 0 \end{aligned} \quad (17)$$

where the parameters satisfy the following positivity constraint $2\vartheta\theta > \nu^2$, and W^2 is another \mathbb{G} -Brownian motion under \mathbb{Q} such that $dW_t^1 dW_t^2 = \rho dt$.

6.2 A PDE based benchmark

For the purpose of testing the accuracy of the Monte Carlo algorithm, we compare the convertible bond prices computed by simulation to the prices obtained with a FD scheme. In particular, as a benchmark, we use the model of Ayache, Forsyth and Vetzal (2003) extended by Yigitbasioglu (2004).

The two factor model in Yigitbasioglu (2004) is exactly equivalent to the model described in the previous section. The price of the convertible bond with call and put features is shown to solve the following Partial Differential Inequality (PDI):

$$\frac{\partial V}{\partial t} + \mathcal{L}V + \lambda \max \left(\mathbf{a}\delta\tilde{S}(t), F\delta^* \right) \begin{array}{l} \geq \\ = \\ \leq \end{array} 0 \quad (18)$$

with either one of the following conditions satisfied at all times in the grid:

for $K^c(t) > \mathbf{a}\tilde{S}(t)$

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}V + \lambda \max(\mathbf{a}\delta\tilde{S}(t), F\delta^*) \leq 0, \\ V - \max(\mathbf{a}\tilde{S}(t), K^p(t)) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}V + \lambda \max(\mathbf{a}\delta\tilde{S}(t), F\delta^*) \geq 0, \\ V - K^c(t) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}V + \lambda \max(\mathbf{a}\delta\tilde{S}(t), F\delta^*) = 0, \\ V - \max(\mathbf{a}\tilde{S}(t), K^p(t)) \geq 0 \\ V - K^c(t) \leq 0 \end{cases}$$

for $K^c(t) \leq \mathbf{a}\tilde{S}(t)$: $V = \mathbf{a}\tilde{S}(t)$

where the spatial operator \mathcal{L} is defined by

$$\begin{aligned} \mathcal{L}V &= \mu\tilde{S}\frac{\partial V}{\partial \tilde{S}} + \frac{1}{2}\sigma^2\tilde{S}^2\frac{\partial^2 V}{\partial \tilde{S}^2} + \vartheta(\theta - r)\frac{\partial V}{\partial r} \\ &+ \frac{1}{2}\nu^2r\frac{\partial^2 V}{\partial r^2} + \rho\sigma\nu\tilde{S}\sqrt{r}\frac{\partial^2 V}{\partial \tilde{S}\partial r} - (r + \lambda)V \end{aligned}$$

A description of the FD scheme that can be used to solve the above PDI can be found in Yigitbasioglu (2004).

7 Numerical Results

In this section, we give a brief account of numerical results obtained with the Monte Carlo algorithm and the FD scheme. In order to test the accuracy of the simulation methodology, we priced more than 10 000 convertible bonds with different maturities, coupon rates, call and put prices as well as different values of default intensity and bond recovery rate. We considered

Table 1: A Short Summary of MC Pricing Errors for a Two-Factor Model

Summary statistics on percentage errors in Monte Carlo prices relative to finite differences across all maturities and coupon rates are reported for different levels of stock returns volatility. The number of CBs in each volatility bracket is given in column two.

Stock Volatility	Number of Priced CBs	Percentage error in MC prices		
		Mean	Max	Min
0.2	3526	0.08	0.40	-0.30
0.4	3494	0.26	0.70	-0.21
0.6	3535	0.41	0.93	-0.75

different parameter values for the short rate process, allowing for flat, upward, as well as downward sloping term structures, with different levels of short rate volatility. Assuming a zero dividend yield, we varied the values of the parameters in the stock price dynamics, allowing for different levels of volatility, and stock recovery rate. Different levels of correlation between the short rate and the stock price were also considered.

The Monte Carlo prices were computed using at most 200,000 simulation paths with 16 exercise times per year. The paths were generated using low-discrepancy Sobol numbers. As a set of basis functions, we used the first three powers of the stock price, the interest rate, their cross product, and a constant. Pricing results are not provided here, but are available from the authors upon request.

Table (1) summarizes the results of the accuracy tests conducted. It appears that the errors tend to be generally small. The average error is 0.24%, while the largest error does not exceed 1% in magnitude.

8 Conclusion

This paper proposes a new methodology for pricing convertible bonds. The methodology is based on the Least-squares Monte Carlo method introduced by Longstaff and Schwartz (2001). The motivation for using Monte Carlo in the context of convertible bond pricing is twofold. First of all, the CB pricing problem is multi-dimensional given the fact that CBs are subject to multiple sources of risk such as equity risk, interest rate risk, volatility, and credit risk. Secondly, many convertible bonds have path-dependent features such as soft calls, resettable conversion ratios, etc.

The proposed methodology takes into account the credit risk by introducing

the time of default as a totally inaccessible stopping time. The stock recovery is modelled as a fraction of the pre-default market value, while the recovery of the CB is assumed to be a fraction of the face value.

In order to apply Monte Carlo, we reformulate the CB pricing problem. Following an approach commonly used in the credit risk literature, we introduce a CB dividend process, and write the CB price as the risk-neutral expectation of the discounted future cash flows when both, the issuer and the investor, follow optimal strategies. Thorough tests of the Monte Carlo algorithm against an accurate Finite Difference scheme have shown that the methodology is reliable.

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