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## Analytic Approximations for Spread Options

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## ABSTRACT

Even in the simple case that two price processes follow correlated geometric Brownian motions with constant volatility no analytic formula for the price of a standard European spread option has been derived, except when the strike is zero in which case the option becomes an exchange option. This paper expresses the price of a spread option as the price of a compound exchange option and hence derives a new analytic approximation for its price and hedge ratios. This approximation has several advantages over existing analytic approximations, which have limited validity and an indeterminacy that renders them of little practical use. Simulations quantify the accuracy of our approach and demonstrate the indeterminacy and inaccuracy of other analytic approximations. The American spread option price is identical to the European option price when the two price processes have identical drifts, and otherwise we derive an expression for the early exercise premium. A practical illustration of the model calibration uses market data on American crack spread options.<sup>1</sup>

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## 1. INTRODUCTION

A spread option is an option whose pay-off depends on the price spread between two correlated underlying assets. If the asset prices are  $S_1$  and  $S_2$  the payoff to a spread option of strike  $K$  is  $[\omega(S_1 - S_2 - K), 0]$  where  $\omega = 1$  for a call and  $\omega = -1$  for a put. Early work on spread option pricing by Ravindran [1993], Shimko [1994], Kirk [1996] assumed each forward price process is a geometric Brownian motion with constant volatility and that these processes have a constant non-zero correlation: we label this the '2GBM' framework for short. The 2GBM assumption allows a simple analytic approximation for the spread option price by reducing the dimension of the uncertainty from two to one. However, as we shall explain below, these approximations suffer from an indeterminacy that renders them practically useless.

The 2GBM framework is tractable but it captures neither the implied volatility smiles that are derived from market prices of univariate options nor the implied correlation smile that is evident from market prices of spread options. In fact correlation 'frowns' rather than 'smiles' are a prominent feature in spread option markets. This is because the pay-off to a spread option decreases with correlation. Hence if market prices of out-of-the-money call and put spread options are higher than the standard 2GBM model prices with constant correlation the implied correlations that are backed out from the 2GBM model will have the appearance of a 'frown'.

Alexander and Scourse [2004] derive approximate analytic prices of European spread options on futures or forward contracts that display both volatility smiles and a correlation frown. They assume the asset prices have a bivariate lognormal mixture distribution and hence obtain prices as a weighted sum of four different 2GBM spread option prices, each of which may be obtained using an analytic approximation such as that of Kirk [1996]. However, most spread options are traded on assets that pay dividends or have carry costs. For instance spread options on equity indices and options on commodity spreads are common. And in most cases the options are American, as is the case for the crack spread options that we consider later in this paper.

Numerical approaches to pricing and hedging spread options that are both realistic and tractable include Carr and Madan [1999] and Dempster and Hong [2000] who advocate models that capture volatility skews on the two assets by introducing stochastic volatility to the price processes. And the addition of price jumps can explain the implied correlation frown, as in the spark spread option pricing model of Carmona and Durrleman [2003a]. However pricing and hedging in this framework necessitates computationally intensive numerical resolution methods such as the fast Fourier transform. Other models provide only a range for spread option prices, as in Durrleman [2001] and Carmona and Durrleman [2005], who provide upper and lower bounds that can be very narrow for certain parameter values. For a detailed survey of these models and a comparison of their performances, the reader is referred to Carmona and Durrleman [2003b]. Recently Li et al. [2006] define a bivariate normal process for the underlying assets and express the price as an expectation of the transformed payoff.

Even retaining the simplicity of the 2GBM framework an exact analytic price for a spread option with non-zero strike is elusive. In this paper we express the price as that of a compound exchange option and thus derive a new analytic approximation for the price and hedge ratios of a spread option. Our approximation always provides a unique and close approximation to the exact price, it is easy to calibrate and hedge ratios are simple to compute. By contrast, other analytic approximations are only valid for spread options of certain strikes and the calibrated option price is not

unique.

The outline of this paper is as follows: Section 2 provides the background to our work, beginning with a summary of the exchange option pricing formula of Margrabe [1978] since this is central to our model. We also provide a derivation of the approximation stated in Kirk [1996] since this is not available in the literature, and we extend the approximation to allow for non-zero dividends or carry costs. Section 3 derives the compound exchange option representation, the analytic approximation to the price and hedge ratios of spread options and remarks on the model calibration. A simulation exercise demonstrates the accuracy of our approximation compared with that of Kirk [1996]. Section 4 extends the framework to accommodate early exercise and here an empirical demonstration of the superiority of our analytic approximation is based on the pricing and hedging performance for American crack spread options traded on the New York Mercantile Exchange (NYMEX) during 2005. The final section summarises our results and concludes.

## 2. BACKGROUND

Here we assume that the two underlying asset prices follow correlated geometric Brownian motions with constant volatilities and constant correlation. We present Margrabe's formula for the price a European exchange option and the approximate pricing formulae for spread options with non-zero strike that are in common use.

### 2.1. Margrabe's Exchange Option Pricing Formula

When the strike of the spread option is zero the option is called an exchange option, since the buyer has the option to exchange one underlying asset for the other. The fact that the strike is zero allows one to reduce the pricing problem to a single dimension, using one of the assets as numeraire. If  $S_{1,t}$  and  $S_{2,t}$  are the spot prices of two assets at time  $t$  then the payoff to an exchange option at the expiry date  $T$  is given by  $[S_{1,T} - S_{2,T}]^+$ . But this is equivalent to an ordinary call option on  $x_t = S_{1,t}/S_{2,t}$  with unit strike.

Assume that the risk-neutral price dynamics are governed by two correlated geometric Brownian motions with constant volatilities given by:

$$dS_{i,t} = (r - q_i)S_{i,t}dt + \sigma_i S_{i,t}dW_{i,t} \quad i = 1, 2 \quad (1)$$

where,  $W_{1,t}$  and  $W_{2,t}$  are Wiener processes under risk neutral measure,  $r$  is the (assumed constant) risk-free interest rate and  $q_1$  and  $q_2$  are the (assumed constant) dividend yields of the two assets. The volatilities  $\sigma_1$  and  $\sigma_2$  are also assumed to be constant as is the returns correlation:

$$\langle dW_{1,t}, dW_{2,t} \rangle = \rho dt$$

Using risk-neutral valuation the price of an exchange option is given by

$$P_t = \mathbb{E}_{\mathbb{Q}} \{ e^{-r(T-t)} [S_{1,T} - S_{2,T}]^+ \} = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ S_{2,T} \max \{ x_T - 1, 0 \} \}$$

where  $x_t$  follows the process

$$dx_t = (q_2 - q_1)x_t dt + \sigma x_t dW_t$$

with

$$dW_t = \rho dW_{1,t} + \sqrt{1 - \rho^2} dW_{2,t}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

Since the both the assets grow at the risk-free rate, the relative drift of  $S_1$  with respect to  $S_2$  due to  $r$  is zero. But as the dividend yields of the assets may be different,  $x_t$  drifts at the rate of  $(q_2 - q_1)$ .

Margrabe [1978] shows that under these assumptions the price  $P_t$  of an exchange option is given by

$$P_t = S_{1,t}e^{-q_1(T-t)}\Phi(d_1) - S_{2,t}e^{-q_2(T-t)}\Phi(d_2) \quad (2)$$

where  $\Phi$  denotes the standard normal distribution function and

$$d_1 = \frac{\ln\left(\frac{S_{1,t}}{S_{2,t}}\right) + (q_2 - q_1 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

## 2.2. Analytic Approximations to Spread Option Prices and Hedge Ratios

In this section we review *analytic* approximations to spread option prices and hedge ratios in the literature rather than the numerical approximation methods described in the introduction. Analytic approximations are preferred over numerical techniques like fast-fourier transform, PDE methods and trees for their computational ease and availability of closed form formulae for hedge ratios.

Kirk [1996] presents an approximate formula for pricing European spread options on futures or forwards. The method extends that of Margrabes' to non-zero but very small strike values. When  $K \ll S_{2,t}$  the displaced diffusion process  $S_{2,t} + K$  can be assumed to be approximately log-normal. Then, the ratio between  $S_{1,t}$  and  $(S_{2,t} + Ke^{-r(T-t)})$  is also approximately log-normal and can be expressed as a geometric Brownian motion process. We outline the main steps of the derivation of the formula in appendix A. Rewrite the pay-off to the European spread option as:

$$[\omega(S_{1,T} - S_{2,T} - K)]^+ = (K + S_{2,T}) \left[ \omega \left( \frac{S_{1,T}}{K + S_{2,T}} - 1 \right) \right]^+$$

$$= (K + S_{2,T}) [\omega(Z_T - 1)]^+$$

where  $\omega = 1$  for a call and  $\omega = -1$  for a put,  $Z_t = \frac{S_{1,t}}{Y_t}$  and  $Y_t = S_{2,t} + Ke^{-r(T-t)}$ . The price  $f_t$  at time  $t$  for a spread option on  $S_1$  and  $S_2$  with strike  $K$ , maturity  $T$  and payoff  $[\omega(S_1 - S_2 - K)]^+$  is given by:

$$f_t = \mathbb{E}_{\mathbb{Q}} \{ Y_t e^{-r(T-t)} \max \{ \omega(Z_T - 1), 0 \} \}$$

$$= \omega \left( S_{1,t} e^{-q_1(T-t)} \Phi(\omega d_{1Z}) - (Ke^{-r(T-t)} + S_{2,t}) e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \Phi(\omega d_{2Z}) \right) \quad (3)$$

where

$$d_{1Z} = \frac{\ln(Z_t) + (r - \bar{r} + \bar{q}_2 - q_1 + \frac{1}{2}\sigma_t^2)(T-t)}{\sigma_t\sqrt{T-t}}$$

$$d_{2Z} = d_{1Z} - \sigma_t\sqrt{T-t}$$

$$\sigma_t = \sqrt{\sigma_1^2 + \sigma_2^2 \left(\frac{S_{2,t}}{Y_t}\right)^2 - 2\rho\sigma_1\sigma_2 \left(\frac{S_{2,t}}{Y_t}\right)}$$

A slightly modified representation of Kirk's formula is

$$P_t^* = \frac{P_t}{Ke^{-rT} + S_{2,t}} = \omega \left( Z_t e^{-q_1(T-t)} \Phi(\omega d_{1Z}) - e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \Phi(\omega d_{2Z}) \right) \quad (4)$$

This representation reduces the dimension of the pricing problem from two to one, which is useful when we extend the formula to price American spread options in section 4.

The price hedge ratios for Kirk's approximation are straightforward to derive from equation (3). Let  $\Delta_x^z$  denotes the delta of  $y$  with respect to  $x$  and  $\Gamma_{xy}^z$  denote the gamma of  $z$  with respect to  $x$  and  $y$ . The two deltas and pure gammas are quite similar to that of Black-Scholes':

$$\Delta_{S_1}^f = \omega e^{-q_1(T-t)} \Phi(\omega d_{1Z})$$

$$\Delta_{S_2}^f = -\omega e^{-q_1(T-t)} \Phi(\omega d_{2Z})$$

$$\Gamma_{S_1 S_1}^f = e^{-q_1(T-t)} \frac{\phi(d_{2Z})}{S_{1,t} \sigma_t \sqrt{T-t}}$$

$$\Gamma_{S_2 S_2}^f = e^{-q_1(T-t)} \frac{\phi(d_{2Z})}{(Ke^{-r(T-t)} + S_{2,t}) \sigma_t \sqrt{T-t}}$$

The cross gamma, i.e., the second order derivative of price with respect to both the underlying assets is given by

$$\Gamma_{S_1 S_2}^f = -e^{-q_1(T-t)} \frac{\phi(d_{1Z})}{(Ke^{-r(T-t)} + S_{2,t}) \sigma_t \sqrt{T-t}} = -e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \frac{\phi(d_{2Z})}{S_{1,t} \sigma_t \sqrt{T-t}}$$

Under the 2GBM assumption other price approximations can be derived that are similar to Kirk's approximation in that they reduce the dimension of the uncertainty from two to one<sup>2</sup>. For instance let  $S_t = S_{1,t}e^{-q_1(T-t)} - S_{2,t}e^{-q_2(T-t)}$  and choose an arbitrary  $M \gg \max\{S_t, \sigma_t\}$ . Then an analytic spread option price approximation based on an approximate lognormal distribution for  $M + S_t$  is:

$$f_t = (M + S_t)\Phi(d_{1M}) - (M + K)e^{-r(T-t)}\Phi(d_{2M})$$

where

$$d_{1M} = \frac{\ln\left(\frac{M+S_t}{M+K}\right) + (r - q + \frac{1}{2}\sigma_t^2)(T-t)}{\sigma_t\sqrt{T-t}}$$

$$d_{2M} = d_{1M} - \sigma_t\sqrt{T-t}$$

$$\sigma_t = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} / (M + S_t)$$

<sup>2</sup>see Eydeland and Wolyniec [2003]

To avoid arbitrage a spread option must be priced consistently with the prices of options on  $S_1$  and  $S_2$ . This implies setting  $\sigma_i$  in (1) equal to the implied volatility of  $S_i$  for  $i = 1, 2$ . Then the implied correlation is calibrated by equating the model and market prices of the spread option. Although the 2GBM model assumes constant volatility the market implied volatilities are not constant with respect to strike. So the strikes  $K_1$  and  $K_2$  at which the implied volatilities  $\sigma_1$  and  $\sigma_2$  are calculated have a significant influence on the results. In Kirk's approximation there is an indeterminacy that arises from the choice of strike for the single asset implied volatilities and the calibrated value of the implied correlation of a spread option with strike  $K$  will not be independent of this choice.

The problem with spread option price approximations such as Kirk's is that the implied volatility and correlation parameters are ill-defined. There are infinitely many pairs  $(S_{1,t}, S_{2,t})$  for which  $S_{1,t} - S_{2,t} = K$  and hence very many possible choices of  $K_1$  and  $K_2$ . Similarly there are infinitely many combinations of market implied  $\sigma_1$ ,  $\sigma_2$ , and  $\rho$  that yield the same  $\sigma$  in equation (3). Hence the construction does not lead to a unique price for the option. To calibrate the model some 'convention' needs to be applied. We have tried using the single asset's at-the-money (ATM) forward volatility to calibrate spread options of all strikes, and several other conventions. None of these gave reasonable results. This points to another major drawback of the approximations: for large values of  $K$ , the log-normality approximation does not hold and neither do the assumptions of constant drift and volatility parameters. Hence the formulae have limited validity. To our knowledge we do not know of any approximation other than ours that is free of a strike convention.

### 3. SPREAD OPTIONS AS COMPOUND EXCHANGE OPTIONS

In this section we derive a representation of the price of a spread option as the sum of the prices of two exchange options. These exchange options are: to exchange a call on one asset with a call on the other asset, and to exchange a put on one asset with a put on the other asset. We retain the 2GBM assumption since our purpose is to compare our approximation with Kirk's approximation. Our approximation arises because we assume that the call and put options in the exchange options have constant volatility. The accuracy of the approximate prices is quantified by simulation.

#### 3.1. The Compound Exchange Option Representation

Let  $(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  be a filtered probability space, where  $\Theta$  is the set of all possible events  $\theta$  such that  $S_{1,t}, S_{2,t} \in [0, \infty)$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is the filtration produced by the sigma algebra of the price pair  $(S_{1,t}, S_{2,t})_{t \geq 0}$  and  $\mathbb{Q}$  is a bivariate risk neutral probability measure and

$$\begin{aligned} \mathcal{L} &= \{\theta \in \Theta : \omega (S_{1,T} - S_{2,T} - K) \geq 0\} \\ \mathcal{A} &= \{\theta \in \Theta : S_{1,T} - mK \geq 0\} \\ \mathcal{B} &= \{\theta \in \Theta : S_{2,T} - (m-1)K \geq 0\} \end{aligned}$$

The payoff to the spread option of strike  $K$  at time  $T$  can be written

$$\begin{aligned} \mathbf{1}_{\mathcal{L}} \omega [S_{1,T} - S_{2,T} - K] &= \mathbf{1}_{\mathcal{L}} \omega ([S_{1,T} - mK] - [S_{2,T} - (m-1)K]) \\ &= \mathbf{1}_{\mathcal{L}} \omega \left( \mathbf{1}_{\mathcal{A}} [S_{1,T} - mK] - \mathbf{1}_{\mathcal{B}} [S_{2,T} - (m-1)K] \right. \\ &\quad \left. + (1 - \mathbf{1}_{\mathcal{A}}) [S_{1,T} - mK] - (1 - \mathbf{1}_{\mathcal{B}}) [S_{2,T} - (m-1)K] \right) \end{aligned}$$

where  $m$  is any positive real number. Since a European option price at time  $t$  depends only on the terminal price densities, we have

$$\begin{aligned}
 f_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ \omega \mathbf{1}_{\mathcal{L}} [S_{1,T} - S_{2,T} - K] \} \\
 &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \omega \mathbf{1}_{\mathcal{L}} \left( \mathbf{1}_{\mathcal{A}} [S_{1,T} - mK] - \mathbf{1}_{\mathcal{B}} [S_{2,T} - (m-1)K] \right. \right. \\
 &\quad \left. \left. + (1 - \mathbf{1}_{\mathcal{A}}) [S_{1,T} - mK] - (1 - \mathbf{1}_{\mathcal{A}}) [S_{2,T} - (m-1)K] \right) \right\} \\
 &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \omega \left( \mathbf{1}_{\mathcal{L} \cap \mathcal{A}} [S_{1,T} - mK] - \mathbf{1}_{\mathcal{L} \cap \mathcal{B}} [S_{2,T} - (m-1)K] \right. \right. \\
 &\quad \left. \left. + (1 - \mathbf{1}_{\mathcal{L} \cap \mathcal{B}}) [(m-1)K - S_{2,T}] - (1 - \mathbf{1}_{\mathcal{L} \cap \mathcal{A}}) [mK - S_{1,T}] \right) \right\} \\
 &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left( [S_{1,T} - mK]^+ - [S_{2,T} - (m-1)K]^+ \right) \right]^+ \right\} \\
 &\quad + e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left( [(m-1)K - S_{2,T}]^+ - [mK - S_{1,T}]^+ \right) \right]^+ \right\} \\
 &= e^{-r(T-t)} \left( \mathbb{E}_{\mathbb{Q}} \{ [\omega [U_{1,T} - U_{2,T}]]^+ \} + \mathbb{E}_{\mathbb{Q}} \{ [\omega [V_{2,T} - V_{1,T}]]^+ \} \right) \tag{5}
 \end{aligned}$$

where  $U_{1,T}$ ,  $V_{1,T}$  are pay-offs to European call and put options on asset 1 with strike  $mK$  and  $U_{2,T}$ ,  $V_{2,T}$  on asset 2 with strike  $(m-1)K$  respectively. This shows that a spread option is exactly equivalent a compound exchange option (CEO) on two call options, with prices  $U_{1,t}$  and  $U_{2,t}$  and two put options with prices  $V_{1,t}$  and  $V_{2,t}$ .

We now describe the processes of the two call and put options  $U_{i,t}$  and  $V_{i,t}$ ,  $i = 1, 2$ . From (1):

$$\begin{aligned}
 dU_{i,t} &= \frac{\partial U_{i,t}}{\partial t} dt + \frac{\partial U_{i,t}}{\partial S_{i,t}} dS_{i,t} + \frac{1}{2} \frac{\partial^2 U_{i,t}}{\partial S_{i,t}^2} dS_{i,t}^2 \\
 &= \left( \frac{\partial U_{i,t}}{\partial t} + rS_{i,t} \frac{\partial U_{i,t}}{\partial S_{i,t}} + \sigma_i^2 S_{i,t}^2 \frac{\partial^2 U_{i,t}}{\partial S_{i,t}^2} \right) dt + \sigma_i S_{i,t} \frac{\partial U_{i,t}}{\partial S_{i,t}} dW_{i,t} \\
 &= rU_{i,t} dt + \sigma_i S_{i,t} \Delta_{U_{i,t}} dW_{i,t} \tag{6}
 \end{aligned}$$

That is

$$dU_{i,t} = rU_{i,t} dt + \xi_i U_{i,t} dW_{i,t} \tag{7}$$

where

$$\xi_i = \sigma_i \frac{S_{i,t}}{U_{i,t}} \frac{\partial U_{i,t}}{\partial S_{i,t}}$$

Similarly

$$dV_{i,t} = rV_{i,t} dt + \eta_i V_{i,t} dW_{i,t} \tag{8}$$

where

$$\eta_i = \sigma_i \frac{S_{i,t}}{V_{i,t}} \left| \frac{\partial V_{i,t}}{\partial S_{i,t}} \right|$$

### 3.2. A New Analytic Approximation

In this section we make the approximation that  $\xi_i$  and  $\eta_i$  are constant throughout  $[t, T]$  with

$$\begin{aligned}
 \xi_i &= \sigma_i \mathbb{E}_{\mathbb{Q}} \left\{ X_{i,s} \mid \mathcal{F}_t \right\} \frac{\partial U_{i,t}}{\partial S_{i,t}} \\
 \eta_i &= \sigma_i \mathbb{E}_{\mathbb{Q}} \left\{ Y_{i,s} \mid \mathcal{F}_t \right\} \left| \frac{\partial V_{i,t}}{\partial S_{i,t}} \right| \tag{9}
 \end{aligned}$$

where  $X_{i,t} = \frac{S_{i,t}}{U_{i,t}}$ ,  $Y_{i,t} = \frac{S_{i,t}}{V_{i,t}}$ , and  $s \in [t, T]$ .

Under the 2GBM assumption for the spread option's underlying prices the exchange option price distributions will not be lognormal. However they will be approximately lognormal if the option remains deep in-the-money (ITM) or deep out-of-the-money (OTM) until expiry. An intuitive explanation of this is that the price of a deep ITM is a linear function of the relative price of the two underlying assets and under the 2GBM assumption the relative price distribution is lognormal. The price of a deep OTM exchange option is approximately zero.

For a mathematical justification of this assumption, note that  $X_{i,t}$  and  $Y_{i,t}$  are martingales and under the equivalent martingale measure  $\mathbb{Q}$  we have  $X_{i,t} = \mathbb{E}_{\mathbb{Q}} \{X_{i,s} | \mathcal{F}_t\}$  and  $Y_{i,t} = \mathbb{E}_{\mathbb{Q}} \{Y_{i,s} | \mathcal{F}_t\}$ . Hence

$$\sigma_{X_i} = \sqrt{\xi_i^2 + \sigma_i^2 - 2\xi_i\sigma_i} = |\xi_i - \sigma_i|$$

So

$$\begin{aligned} \sigma_{X_i} &= \sigma_i |X_{i,t} \Delta U_{i,t} - 1|, \quad \text{and} \\ \sigma_{Y_i} &= \sigma_i |Y_{i,t} \Delta V_{i,t} + 1| \end{aligned} \quad (10)$$

Hence as  $\sigma_{X_i} \rightarrow 0$ , the net quadratic variation  $\sum_t^T \langle dX_{i,s}, dX_{i,s} \rangle = \sum_t^T \sigma_{X_i}^2 ds \rightarrow 0$  and  $X_{i,s} \approx X_{i,t}$  for  $s \in [t, T]$ . The exchange options that are used to construct our pricing formula do not need to have traded prices. We calibrate the model to the market prices of spread options only. All four exchange options are determined by the single parameter  $m$  and we may choose this to minimise the volatilities of  $X_{i,t}$  and  $Y_{i,t}$  and, as a result, minimize the approximation error. Therefore we express  $\xi_i$  and  $\eta_i$  as in equation (9) and assume them to be constant.

We now apply Margrabe's formula to derive an analytic price for the spread option as the price of a compound exchange option on two calls and puts. The risk neutral price of the spread option at time  $t$  is given by:

$$f_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{[\omega (U_{1,T} - U_{2,T})]^+\} + e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{[\omega (V_{2,T} - V_{1,T})]^+\} \quad (11)$$

so its price may be obtained using Margrabe's formula:

$$f_t = e^{-r(T-t)} \omega [U_{1,t} \Phi(\omega d_{1U}) - U_{2,t} \Phi(\omega d_{2U}) - (V_{1,t} \Phi(-\omega d_{1V}) - V_{2,t} \Phi(-\omega d_{2V}))] \quad (12)$$

where

$$\begin{aligned} d_{1A} &= \frac{\ln \left( \frac{A_{1,t}}{A_{2,t}} \right) + (q_2 - q_1 + \frac{1}{2} \sigma_A^2) (T - t)}{\sigma_A \sqrt{T - t}} \\ d_{2A} &= d_{1A} - \sigma_A \sqrt{T - t} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sigma_u &= \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2} \\ \sigma_v &= \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2} \end{aligned}$$

The prices of options  $U_i$  and  $V_i$  are given by:

$$\begin{aligned} U_{i,t} &= S_{i,t} e^{-q_i(T-t)} \Phi(d_{1i}) - K_i e^{-r(T-t)} \Phi(d_{2i}) \\ V_{i,t} &= K_i e^{-r(T-t)} \Phi(-d_{2i}) - S_{i,t} e^{-q_i(T-t)} \Phi(-d_{1i}) \end{aligned} \quad (14)$$

where

$$\begin{aligned} d_{1i} &= \frac{\ln\left(\frac{S_{i,t}}{K_i}\right) + (r - q_i + \frac{1}{2}\sigma_i^2)(T - t)}{\sigma_i\sqrt{T - t}} \\ d_{2i} &= d_{1i} - \sigma_i\sqrt{T - t} \end{aligned}$$

where  $K_1 = mK$  and  $K_2 = (m - 1)K$ . Hence the four vanilla options that are used in the model calibration are determined by the parameter  $m$  which is calibrated  $m$  to minimize the volatilities of  $X_{i,t}$  and  $Y_{i,t}$ . Under the assumption of complete markets there exist at least two option price pairs  $\{U_{1,t}, U_{2,t}\}$  and  $\{V_{1,t}, V_{2,t}\}$  such that (12) holds. Note that in practice the calibrated value of  $m$  will depend on the strike and maturity of the spread option.

In equation (12) there are two terms on the right hand side, one representing the discounted expected pay-off to the exchange option with pay-off  $[U_{1,T} - U_{2,T}]^+$  and the other representing the discounted expected pay-off to the exchange option with pay-off  $[V_{2,T} - V_{1,T}]^+$ . To see this, note that for a call spread option:

$$\begin{aligned} f_t &= e^{-r(T-t)} \left( U_{1,t} \frac{\Phi(d_{1U})}{\Phi(d_{2U})} - U_{2,t} \right) \Phi(d_{2U}) \\ &\quad + e^{-r(T-t)} \left( V_{2,t} \frac{\Phi(-d_{2V})}{\Phi(-d_{1V})} - V_{1,t} \right) \Phi(-d_{1V}) \end{aligned} \quad (15)$$

where  $\Phi(d_{2U})$  is the risk neutral probability that  $U_{1,T} > U_{2,T}$  and  $U_{1,t} \frac{\Phi(d_{1U})}{\Phi(d_{2U})}$  is the conditional expectation of  $U_{1,T}$  given  $U_{1,T} > U_{2,T}$ . Similarly,  $\Phi(-d_{1V})$  is the risk neutral probability that  $V_{2,T} > V_{1,T}$  and  $V_{2,t} \frac{\Phi(-d_{2V})}{\Phi(-d_{1V})}$  is the conditional expectation of  $V_{2,T}$  given  $V_{2,T} > V_{1,T}$ .

### 3.3. Approximate Price Hedge Ratios

The delta and gamma hedge ratios for our analytic approximation are straightforward to derive by differentiating the model price with respect to each underlying. Again, let  $\Delta_x^z$  denotes the delta of  $z$  with respect to  $x$  and  $\Gamma_{xy}^z$  denote the gamma of  $z$  with respect to  $x$  and  $y$ :

$$\begin{aligned} \Delta_{S_i}^f &= \Delta_{U_i}^f \Delta_{S_i}^{U_i} + \Delta_{V_i}^f \Delta_{S_i}^{V_i} \\ \Gamma_{S_i S_i}^f &= \Gamma_{U_i}^f \left( \Delta_{S_i}^{U_i} \right)^2 + \Gamma_{S_i}^{U_i} \Delta_{U_i}^f + \Gamma_{V_i}^f \left( \Delta_{S_i}^{V_i} \right)^2 + \Gamma_{S_i}^{V_i} \Delta_{V_i}^f \\ \Gamma_{S_1 S_2}^f &= \Gamma_{S_2 S_1}^f = \Gamma_{U_1 U_2}^f \Delta_{S_1}^{U_1} \Delta_{S_2}^{U_2} + \Gamma_{V_1 V_2}^f \Delta_{S_1}^{V_1} \Delta_{S_2}^{V_2} \end{aligned} \quad (16)$$

The CEO model Greeks given by equations (16) are better approximations than those derived from Kirk's formula for the reasons discussed in the previous section. So other approximation methods can lead to substantial hedging errors as well as inaccurate pricing.

### 3.4. Hedging Volatility and Correlation Risks

Spread options may be delta-gamma hedged by taking positions in the underlying assets and options on these. But hedging volatility and correlation is more complicated. In this next section we derive an expression for the spread option price sensitivity to correlation.

We remark that other analytic approximations yield correlation sensitivities that are proportional to the option vega because they are all based on a volatility of the form:

$$\sigma = \sqrt{\omega_1\sigma_1^2 + \omega_2\sigma_2^2 - 2\omega_3\rho\sigma_1\sigma_2}$$

where all the terms on the right hand side are constant. Hence the sensitivity of volatility to correlation is constant, and this implies that the option price's correlation sensitivity is just a constant times the option vega. Also the model implied correlation is not clearly defined. Correlation is merely calibrated as a free parameter independent of the spread option strike and the underlying volatilities. Thus market sentiments such as the correlation frown are not captured by the model. But then it is meaningless to hedge the spread option correlation based on the calibrated values of model parameters. Moreover, volatility hedging is complicated by the fact that one is likely to hedge the volatilities with the wrong options if the strike convention is not chosen correctly. Theoretically there are infinitely many possible strikes for the two vanilla options and the strikes are chosen without relating them to vega risks. Hence the hedging errors accrued from incorrect vega hedging along with every other unhedged risk are attributed to correlation risk. Clearly these models fail to quantify correlation risks accurately and this is likely to have a serious effect on the P&L of the hedging portfolio.

We now structure the CEO model so that the implied correlation  $\rho$  is directly related to  $m$ , the only independent and therefore central parameter. The vanilla option implied volatilities and the exchange option volatilities are then also determined by  $m$ . This construction provides a closed form formula for the sensitivity of the spread option price to correlation. In other words the correlation smile or frown is endogenous to the model.

In the following we write the spread option price as  $f = f(U_1, U_2, \sigma_u, V_1, V_2, \sigma_v)$  where:

$$\begin{aligned}\sigma_u &= \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2} \\ \sigma_v &= \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2}\end{aligned}$$

Here  $\sigma_u$  and  $\sigma_v$  are the volatilities of the exchange options on calls and puts respectively. The sensitivity of the option price with respect to correlation is thus:

$$\frac{df}{d\rho} = \frac{df}{dm} \frac{dm}{d\rho}$$

where

$$\begin{aligned}\frac{df}{dm} &= \frac{\partial f}{\partial U_1} \frac{dU_1}{dm} + \frac{\partial f}{\partial U_2} \frac{dU_2}{dm} + \frac{\partial f}{\partial \sigma_u} \frac{d\sigma_u}{dm} \\ &\quad + \frac{\partial f}{\partial V_1} \frac{dV_1}{dm} + \frac{\partial f}{\partial V_2} \frac{dV_2}{dm} + \frac{\partial f}{\partial \sigma_v} \frac{d\sigma_v}{dm}\end{aligned}$$

The above equation shows that the vegas of the spread option affect the correlation sensitivity. By contrast with other analytic approximations, in the CEO model the volatility and correlation hedge ratios may be independent of each other. We set  $\frac{d\sigma_u}{d\rho} = 0$  and  $\frac{d\sigma_v}{d\rho} = 0$ . In other words, we choose  $m$  and  $\rho$  so that the volatility of the spread option is invariant to changes in correlation. We call this volatility the *correlation invariant volatility* (CIV).

The total derivative of  $\sigma_u$  is:

$$d\sigma_u = \frac{\partial\sigma_u}{\partial\tilde{\xi}_1}d\tilde{\xi}_1 + \frac{\partial\sigma_u}{\partial\tilde{\xi}_2}d\tilde{\xi}_2 + \frac{\partial\sigma_u}{\partial\rho}d\rho$$

Hence

$$\begin{aligned} \frac{d\sigma_u}{d\rho} &= \frac{\partial\sigma_u}{\partial\tilde{\xi}_1} \frac{d\tilde{\xi}_1}{dm} \frac{dm}{d\rho} + \frac{\partial\sigma_u}{\partial\tilde{\xi}_2} \frac{d\tilde{\xi}_2}{dm} \frac{dm}{d\rho} + \frac{\partial\sigma_u}{\partial\rho} \\ &= \mathbb{A} \frac{dm}{d\rho} - \frac{\tilde{\xi}_1\tilde{\xi}_2}{\sigma_u} \end{aligned} \quad (17)$$

where<sup>3</sup>

$$\mathbb{A} = \frac{1}{\sigma_u} \left( \frac{d\tilde{\xi}_1}{dm} (\tilde{\xi}_1 - \rho\tilde{\xi}_2) + \frac{d\tilde{\xi}_2}{dm} (\tilde{\xi}_2 - \rho\tilde{\xi}_1) \right)$$

Similarly

$$\frac{d\sigma_v}{d\rho} = \mathbb{B} \frac{dm}{d\rho} - \frac{\eta_1\eta_2}{\sigma_v} \quad (18)$$

where

$$\mathbb{B} = \frac{1}{\sigma_v} \left( \frac{d\eta_1}{dm} (\eta_1 - \rho\eta_2) + \frac{d\eta_2}{dm} (\eta_2 - \rho\eta_1) \right)$$

Now equation (17) implies that

$$\begin{aligned} \frac{dm}{d\rho} &= \frac{1}{\mathbb{A}} \frac{\tilde{\xi}_1\tilde{\xi}_2}{\sigma_u} \\ &= \frac{1}{\tilde{\xi}_1\tilde{\xi}_2 \left( \frac{d\tilde{\xi}_1}{dm} (\tilde{\xi}_1 - \rho\tilde{\xi}_2) + \frac{d\tilde{\xi}_2}{dm} (\tilde{\xi}_2 - \rho\tilde{\xi}_1) \right)} \\ &= \frac{1}{\tilde{\xi}_1\tilde{\xi}_2 \left( \left( \frac{d\tilde{\xi}_1}{dm} \tilde{\xi}_1 + \frac{d\tilde{\xi}_2}{dm} \tilde{\xi}_2 \right) - \rho \left( \frac{d\tilde{\xi}_1}{dm} \tilde{\xi}_2 - \frac{d\tilde{\xi}_2}{dm} \tilde{\xi}_1 \right) \right)} \\ &= g(\tilde{\xi}_1, \tilde{\xi}_2, \rho)^{-1} \end{aligned} \quad (19)$$

Similarly:

$$\begin{aligned} \frac{dm}{d\rho} &= \frac{1}{\eta_1\eta_2 \left( \left( \frac{d\eta_1}{dm} \eta_1 + \frac{d\eta_2}{dm} \eta_2 \right) - \rho \left( \frac{d\eta_1}{dm} \eta_2 - \frac{d\eta_2}{dm} \eta_1 \right) \right)} \\ &= g(\eta_1, \eta_2, \rho)^{-1} \end{aligned} \quad (20)$$

We then have

$$\frac{df}{dm} = K \left( f_{u_1} \frac{\partial U_1}{\partial K_1} + f_{u_2} \frac{\partial U_2}{\partial K_2} + f_{v_1} \frac{dV_1}{dK_1} + f_{v_2} \frac{dV_2}{dK_2} \right) \quad (21)$$

<sup>3</sup>The first order derivatives of  $\tilde{\xi}_i$  and  $\eta_i$  with respect to  $m$  can be calculated from their respective implied volatilities  $\sigma_1$  and  $\sigma_2$  either numerically or by assuming a quadratic (or cubic spline) function of their strikes.

$$\sigma_i = a_i K_i^2 + b_i K_i + c_i \quad \text{where } i = 1, 2$$

and  $a_i$ ,  $b_i$ , and  $c_i$  are some constants that can be estimated using curve fitting methods.

and

$$\begin{aligned} \frac{df}{d\rho} &= \frac{df}{dm} \frac{dm}{d\rho} \\ &= Kg(\xi_1, \xi_2, \rho)^{-1} \left( f_{U_1} \frac{dU_1}{dK_1} + f_{U_2} \frac{dU_2}{dK_2} \right) + Kg(\eta_1, \eta_2, \rho)^{-1} \left( f_{V_1} \frac{dV_1}{dK_1} + f_{V_2} \frac{dV_2}{dK_2} \right) \end{aligned} \quad (22)$$

### 3.5. Calibration

Our calibration problem reduces to calibrating a single parameter  $m$  for each spread option by equating market prices of the spread options to (12). Then the single asset options' strikes are determined because  $K_1 = mK$  and  $K_2 = (m - 1)K$ . And the implied correlation between the options is also determined: since the options follow the same Wiener processes as the underlying prices, their implied correlation is the same.

Let  $f_M$  be the market price of the spread option and  $f$  be the price of a spread option given by equation (12). Then the calibration problem reduces to the following optimisation problem:

$$\min ||f_M - f(m, \rho)|| \quad (23)$$

such that, at a given iteration  $j$ :

1.  $m_j$  satisfies the equation :  $g(\xi_1, \xi_2, \rho_j) - g(\eta_1, \eta_2, \rho_j) = 0$
2.  $\left. \frac{dm}{d\rho} \right|_j = \frac{1}{g(\xi_1, \xi_2, \rho_j)}$
3.  $||\sigma_{x_i}|| + ||\sigma_{y_i}||$  is a minimum

where  $g(x, y, z) = \frac{1}{xy} \left( \left( \frac{\partial x}{\partial m} x + \frac{\partial y}{\partial m} y \right) - z \left( \frac{\partial x}{\partial m} y - \frac{\partial y}{\partial m} x \right) \right)$ .

The above problem can be solved using a one-dimensional gradient method. The first order differential of  $f$  with respect to  $\rho$  is given by

$$\frac{df}{d\rho} = Kg(\xi_1, \xi_2, \rho)^{-1} \left( \frac{\partial f}{\partial U_1} \frac{dU_1}{dK_1} + \frac{\partial f}{\partial U_2} \frac{dU_2}{dK_2} + \frac{\partial f}{\partial V_1} \frac{dV_1}{dK_1} + \frac{\partial f}{\partial V_2} \frac{dV_2}{dK_2} \right) \quad (24)$$

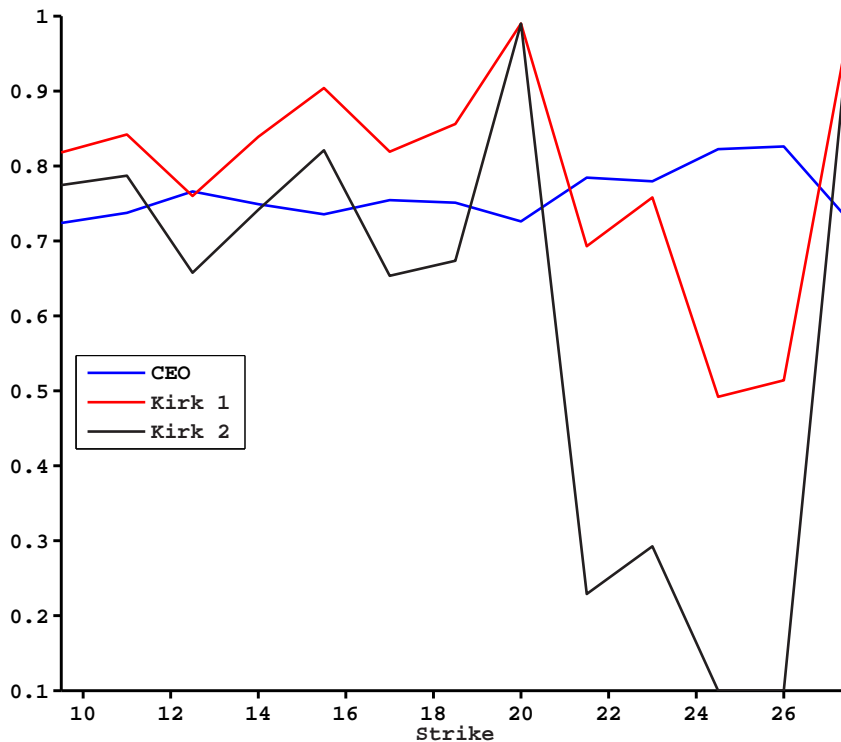
### 3.6. Comparison with Kirk's Approximation

In this section we calibrate our model to simulated spread option prices and compare the calibration errors with those derived from Kirk's approximation. We have used prices  $S_1 = 65$  and  $S_2 = 50$ , and spread option strikes ranging between 9.5 and 27.5 with a step size of 1.5 and maturity 30 days. The spread option prices were simulated using quadratic local volatility and local correlation functions that are assumed to be dependent only on the price levels of the underlying assets and not on time. The at-the-money volatilities were both 30% and the at-the-money forward correlation was 0.80.

Figure 1 compares the implied correlations calibrated from the compound exchange option formula with those obtained from Kirk's approximation. The results illustrate the poor performance

of Kirk's approximation for high strike values. Using Kirk's approximation the root mean square percentage calibration error (RMSE), i.e. where each error is expressed as a percentage of the option price, was 9% using the strike convention and 9.3% using the constant ATM volatility to determine  $\sigma_1$  and  $\sigma_2$ . By contrast the exchange option model's pricing errors are extremely small (the RMSE was 0.53%) and the implied correlation values in figure 1 show greater stability.

FIGURE 1: *Implied Correlations from Kirk's and CEO Approximations*  
 Kirk 1 implied volatilities are calculated using  $K_1 = S_{1,0} - \frac{K}{2}$   
 and  $K_2 = S_{2,0} + \frac{K}{2}$  and Kirk 2 uses ATM constant volatility



This simulation exercise illustrates the main problem with other price approximations for spread options. When we apply a convention for fixing the strikes of the implied volatilities  $\sigma_1, \sigma_2$ , take the implied volatilities from the single asset option prices and then calibrate the implied correlation to the spread option price we obtain totally unrealistic results except for options with very low strikes. For high strike options the model's lognormality assumption is simply not valid.<sup>4</sup>

#### 4. PRICING AND HEDGING AMERICAN SPREAD OPTIONS

In this section derive the early exercise premium for a spread option and extend both Kirk's approximation and our approximation to pricing American spread options. The price of American

<sup>4</sup>Attempts to use the exchange option strike convention with  $K_1 = m(K, T)K$  and  $K_2 = (m(K, T) - 1)K$  led to even greater pricing errors. A possible 'quick fix' could be to change the strike convention so that it can be different for each spread option, but this is very ad hoc.

style options on single underlying assets is mainly determined by the type of the underlying asset, the prevailing discount rate, and the presence of any dividend yield. The option to exercise early suggests that these options are more expensive than their European counterparts but there are many instances when it is not optimal to exercise an option early. American calls on non-dividend paying stocks and calls or puts on forward contracts are two examples where it is never optimal to exercise the option early (see James [2003]). Since no traded options are perpetual the expiry date forces the price of American options to converge to the price of their European counterparts. Before expiry, the prices of American calls and puts are always greater than or equal to the corresponding European calls and puts.

#### 4.1. The Early Exercise Premium of a Spread Option

In the free boundary pricing methods of McKean [1965], Kim [1990], Carr et al. [1989], Jacka [1991], and others the price of an American option with payoff  $[\omega(S_t - K)]^+$  on one underlying asset with price process (1) is given by:

$$P(S_t, t) = P^E(S_t, T, \omega) + \omega \int_t^T q S_t e^{-q(s-t)} \Phi(\omega d_1(S_t, B_t, s-t)) ds - \omega \int_t^T r K e^{-r(s-t)} \Phi(\omega d_2(S_t, B_t, s-t)) ds \quad (25)$$

where  $\omega = 1$  for a call and  $-1$  for a put and  $B_t$  is the early exercise boundary.

In the case of multiple underlying assets the behaviour of American options is similar to that of single asset American options, with some notable exceptions. Rubinstein [1991] was the first to note that an American exchange option is equivalent to a standard option in a modified yet equivalent financial market. The problem pricing reduces to that of pricing a plain vanilla option by taking one of the assets as the numeraire instead of the money market account with the corresponding equivalent martingale measure. Then the prices of such claims can be found using the early exercise premium (EEP) representation (see Detemple [2005]).

We can express the price of an American spread option as a sum of its European counterpart and an early exercise premium. Consider the simple case of an American exchange option on two assets. Let  $S_{1,t}$  and  $S_{2,t}$  be the prices of two assets at time  $t$  given by equation (1) and  $x_t = \frac{S_{1,t}}{S_{2,t}}$ , as in section 2.1. Then the payoff to an exchange option at maturity  $T$  is  $S_{2,T}[x_T - 1, 0]$ .

Let  $P_t^E$  and  $P_t^A$  be the prices of a European and American exchange option respectively. The EEP representation gives

$$P_t^A(x_t) = P_t^E(x_t) + \int_t^T q_1 x_t e^{-q_1(s-t)} \Phi(d_1(x_t, B_s, s-t, q_1, q_2, \sigma_x)) ds - \int_t^T q_2 e^{-q_2(s-t)} \Phi(d_2(x_t, B_s, s-t, q_1, q_2, \sigma_x)) ds \quad (26)$$

where,

$$d_1(x_t, B_t, T-t, q_1, q_2, \sigma_x) = \frac{\ln\left(\frac{x_t}{B_t}\right) + (q_2 - q_1 + \frac{1}{2}\sigma_x^2)(T-t)}{\sigma_x \sqrt{T-t}}$$

$$d_2(x_t, B_t, T-t, q_1, q_2, \sigma_x) = d_1(x_t, B_t, T-t, q_1, q_2, \sigma_x) - \sigma_x \sqrt{T-t}$$

and  $\sigma_x$  is as defined in section 2.1. This shows that for the early exercise premium to be positive we require  $q_1 > 0$ .<sup>5</sup>

Now consider the case when the two underlying assets are futures contracts. Since futures do not have dividends, the above equation implies  $P_t^A = P_t^E$ . Hence whilst an American option on a single futures contract may be worth more than the corresponding European option this is not necessarily the case for American options on multiple assets. Broadie and Detemple [1997] provide a detailed discussion of pricing American options on two assets stating properties of the exercise region and giving a recursive integral equation which is satisfied by the early exercise boundary. At present there are no efficient methods available to calculate the early exercise boundary in the two asset case. However in the following sections we reduce the dimension of the problem to one

## 4.2. Extension of Kirk's Formula

In section 2.1 the random variable  $Z$  was approximately log normal for small  $K$  values and this allowed one to express the price of a European put spread as that of an ordinary European put. By the same construction we can use  $Z$  to express the price of an American put spread as an ordinary American put on  $Z$  with strike 1. The intrinsic value of the option at time  $t$  is given by,

$$[Ke^{-r(T-t)} - S_{1,t} + S_{2,t}]^+ = [Y_t - S_{1,t}]^+$$

The above resembles the payoff of an exchange option written on  $Y_t$  and  $S_{1,t}$ , and both processes are observable in the market. Recalling equations (1) and (32) we have,

$$\begin{aligned} P_t^A(Z_t) = P_t^*(Z_t) & - \int_t^T q_1^* Z_t e^{-q_1^*(s-t)} \Phi(-d_1(Z_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \\ & + \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(Z_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \end{aligned} \quad (27)$$

where  $q_1^* = q_1$  and  $q_2^* = (r - \bar{r} + \bar{q}_2)$ .

At the early exercise boundary, i.e., when  $Z_t = B_t$ , the price given by equation (27) equals  $1 - B_t$ .

$$\begin{aligned} 1 - B_t = P_t^*(B_t) & - \int_t^T q_1^* B_t e^{-q_1^*(s-t)} \Phi(-d_1(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \\ & + \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \end{aligned} \quad (28)$$

This is the value match condition. Moreover, at  $B_t$  the slope of the price curve of equation (27) is that of  $1 - B_t$ . This is called as the high contact condition and it can be obtained by differentiating equation (28) with respect to  $B_t$ , giving:

$$\begin{aligned} \frac{\partial P_t^*(B_t, 1, T-t)}{\partial B_t} - 1 & = \frac{\partial}{\partial B_t} \left( \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \right) \\ & - \frac{\partial}{\partial B_t} \left( \int_t^T q_1^* B_t e^{-q_1^*(s-t)} \Phi(-d_1(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) ds \right) \end{aligned}$$

<sup>5</sup>If  $q_1 = 0$  and  $q_2 > 0$  then  $P_t^A < P_t^E$ .

### 4.3. Extension of Compound Exchange Option Formula

In section 3 we showed how a European spread option price is equivalent to the price of an exchange option on two deep in-the-money call options. We may choose  $m$  to be sufficiently small so that the option price processes closely imitate that of underlying assets and hence carry costs or dividends on the underlying assets can alter the prices of these in-the-money options considerably. Therefore any change in price of the underlying assets due to dividends or carry costs must be accounted for when pricing a spread option as a compound exchange option.

Consider the price process of each underlying asset and the corresponding call options. The solutions to their stochastic differential equations at time  $t$  are given by,

$$S_{i,T} = S_{i,t} e^{(r-q_i-\frac{1}{2}\sigma_i^2)(T-t)+\sigma_i dW_i}$$

$$U_{i,T} = U_{i,t} e^{(r-q_i^*-\frac{1}{2}\xi_i^2)(T-t)+\xi_i dW_i}$$

Dividing  $S_{i,T}$  by  $U_{i,T}$  and using the approximation  $\xi_i \approx \sigma_i$ , to eliminate the stochastic term

$$q_i^* = \frac{1}{T} \left( \ln \left( \frac{S_{i,T}}{S_{i,0}} \right) - \ln \left( \frac{U_{i,T}}{U_{i,0}} \right) \right) + q_i$$

We now rewrite equations (7) and (8) as:

$$\begin{aligned} dU_{i,t} &= (r - q_i^*)U_{i,t}dt + \xi_i U_{i,t}dW_i \\ dV_{i,t} &= (r - q_i^*)V_{i,t}dt + \xi_i V_{i,t}dW_i \end{aligned} \quad (29)$$

where  $q_1^*$  and  $q_2^*$  are the equivalent dividend yields of the options.

It should be noted that even though we shall be pricing American spread options, the two call options with prices  $U_1$  and  $U_2$  remain European style options. Although the exchange option may be exercised before maturity, the call options may be exercised only at expiry. Since the compound exchange option replicates the cash flow of a spread option, when exercised they will yield the same payoff. Since most of the trades are cash settled this is adequate. Even in commodity markets where the options are exercised by the physical delivery of goods, this adjustment can be justified as the underlying future contracts' expiry date is the same as or later than that of the spread option.

Let us now restrict our analysis to the case that there are no dividend yields or carry costs, such as when the underlying assets are future contracts. We now price an American spread option as an American compound exchange option using the early exercise premium representation given by equation (26). Define martingale processes  $X_t = \frac{U_{1,t}}{U_{2,t}}$  and  $Y_t = \frac{V_{2,t}}{V_{1,t}}$ . Then American spread option price is given by

$$\begin{aligned} f_t^A &= f_t^E(X_t, \omega) + \omega \left( \int_t^T q_1^* X_t e^{-q_1^*(s-t)} \Phi(\omega d_1(X_t, B_s, s-t, q_1^*, q_2^*, \sigma_X)) ds \right. \\ &\quad \left. - \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(\omega d_2(X_t, B_s, s-t, q_1^*, q_2^*, \sigma_X)) ds \right) \\ &\quad + f_t^E(Y_t, \omega) + \omega \left( \int_t^T q_1^* Y_t e^{-q_1^*(s-t)} \Phi(\omega d_1(Y_t, B_s, s-t, q_1^*, q_2^*, \sigma_Y)) ds \right. \\ &\quad \left. - \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(\omega d_2(Y_t, B_s, s-t, q_1^*, q_2^*, \sigma_Y)) ds \right) \end{aligned} \quad (30)$$

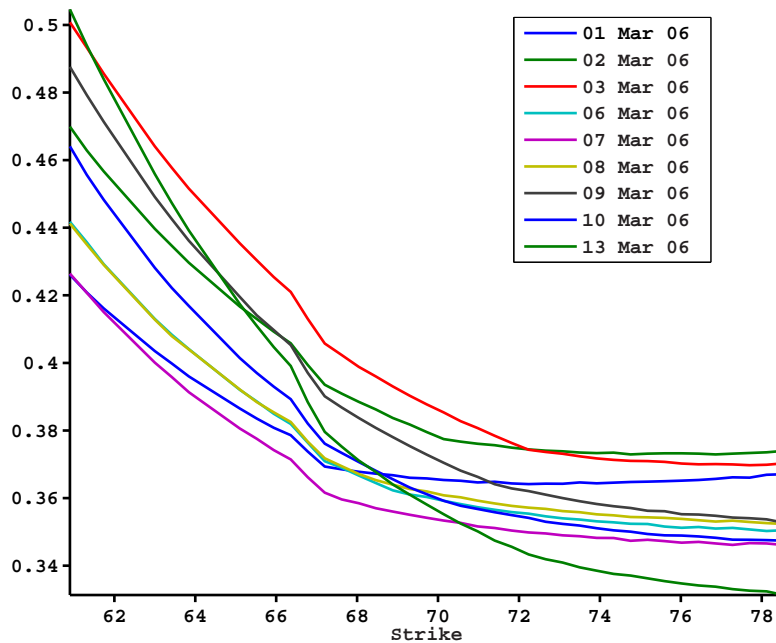
and hence American spread options on futures or non dividend paying stocks are worth the same as their European counterparts.

#### 4.4. Empirical Results

We now test the pricing performance of the exchange option approximation using 1:1 American crack spread option data traded at NYMEX between September 2005 and May 2006. The crack spread options are on gasoline - crude oil and are traded on the price differential between the futures contracts of WTI light sweet crude oil and gasoline. Option data for American style contracts on each of these individual futures contracts were also obtained for the same time period along with the futures prices. The size of all the futures contracts is 1000 bbls.

Figures 2 and 3 depict the implied volatility skews in gasoline and crude oil on several of the days during the sample period. These pronounced negative implied volatility skews indicate that a suitable pricing model should exhibit a positive skew in implied correlation as a function of the spread option strike.

FIGURE 2: *Implied Volatility of Gasoline*



We compare the results of Kirk's approximation with the exchange option approximation by calibrating each model to the market prices of the gasoline - crude oil crack spread over consecutive trading dates starting from 1st March 2006 to 15th March 2006, these being days of particularly high trading volumes. From figure 4 we can clearly see that Kirk's approximation gives an error that increases drastically for high strike values, as was also the case in our simulation results. On the other hand the compound exchange option model errors were found to be close to zero for all strikes on all dates. Figure 5 shows that the implied correlations that are calibrated from the compound exchange option approximation exhibit a realistic, positively sloped skew on each day of the sample. However, the implied correlations computed from Kirk's approximation were found to be equal to 0.99 for all strikes and on every day.

FIGURE 3: Implied Volatility of Crude Oil

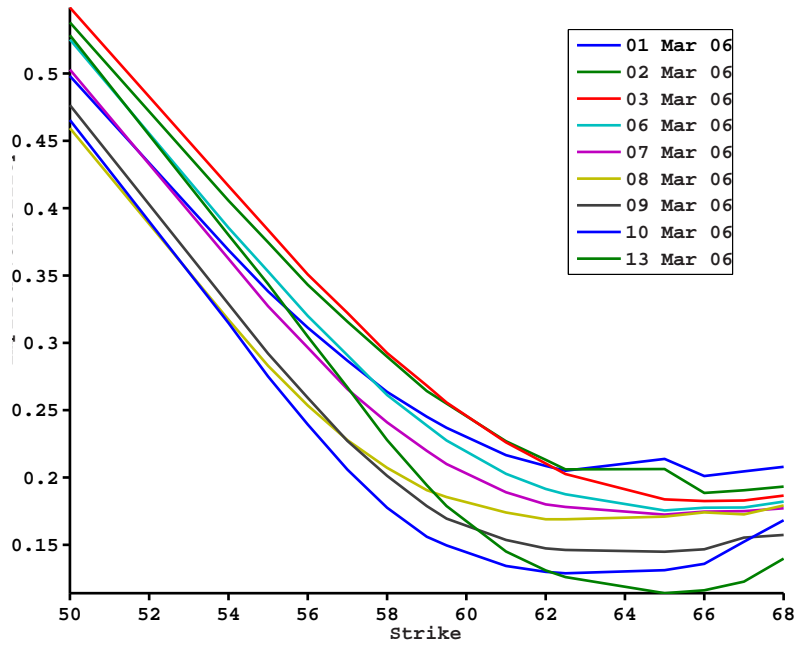


FIGURE 4: Kirks and CEO Pricing Errors

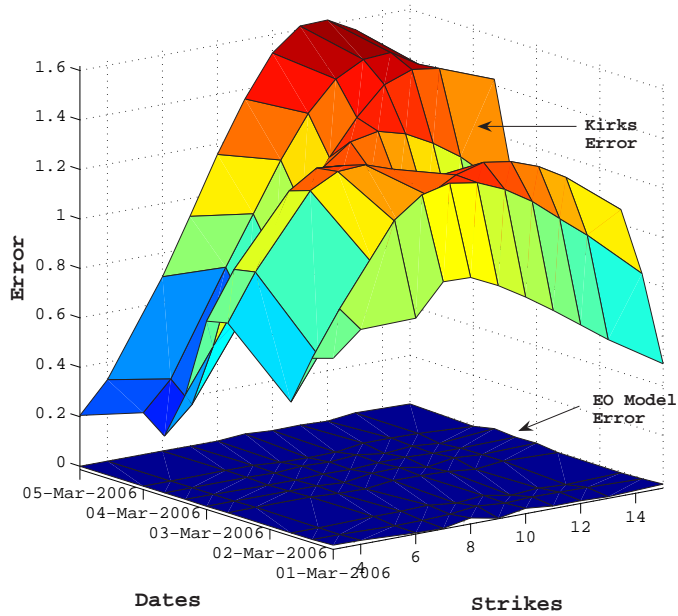
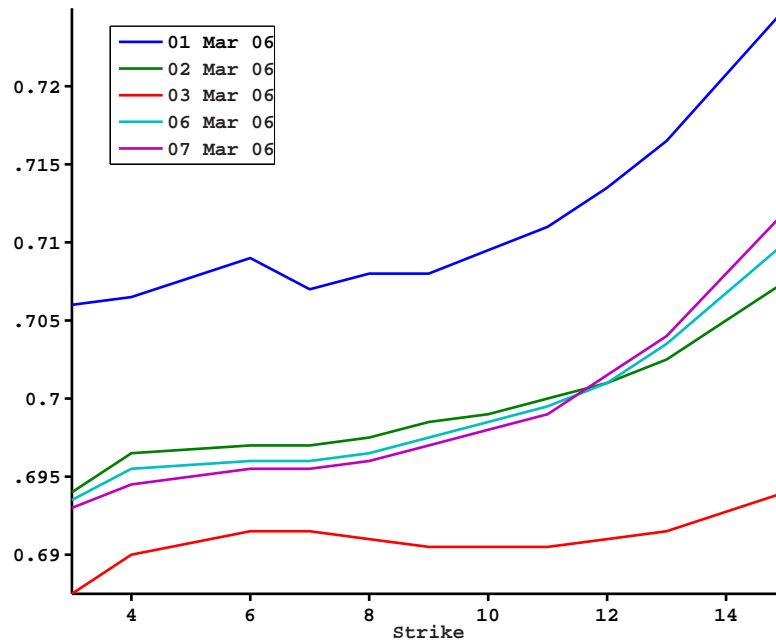


FIGURE 5: *Implied Correlation Skews of CEO Model*



Figures 6 and 7 compare the two deltas and gammas of each model, calibrated on 1st March 2006 and depicted as a function of the spread option strike. The same features are evident on all other days in the sample: at every strike the exchange option delta is much smaller than the delta that is obtained through Kirk’s formula. Similar remarks apply to the gamma hedges, particularly for the gamma hedge on crude oil. We conclude that the use of Kirk’s approximation may lead to significant over hedging.

## 5. CONCLUSION

This paper highlights certain difficulties with pricing and hedging spread options based on approximations such as that of Kirk [1996]. There are two substantial problems: the approximation is only valid for spread options with low strikes and an arbitrary strike convention is necessary to determine the implied volatilities in the calibration. Thus the approximate prices and hedge ratios only apply to spread options with very low strikes and even these have questionable accuracy, since their values depend on the ad hoc choice of strike convention. We have tested several strike conventions for fixing the implied volatilities of the single asset options but in each case their market prices are inconsistent with the market prices of spread options, except for spread options with very low strikes. Moreover, for the crack spread option data all choices of strike convention yielded almost constant correlations that were very close to 1, which is unrealistic.

By contrast, the compound exchange option approximation provides accurate prices at all strikes and realistic values for implied correlation. Other advantages of the compound exchange option approximation are the ease of calibration and the simple computation of the option’s price sensi-

FIGURE 6: Delta with respect to Gasoline (left) and Crude Oil (right)

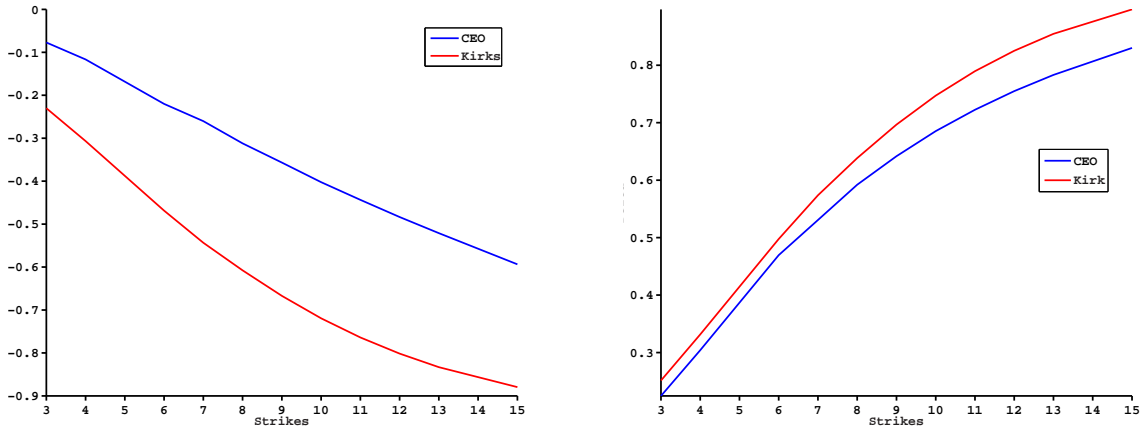
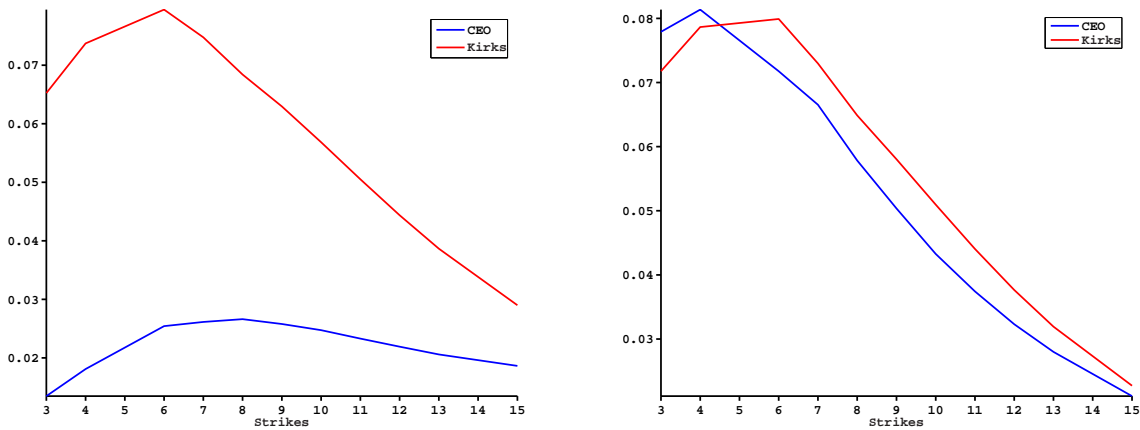


FIGURE 7: Gamma with respect to Gasoline (left) and Crude Oil (right)



tivities. We have found empirically that the compound exchange option approach specifies deltas and gammas that are much smaller than the deltas and gammas from Kirk's approximation and we thus have reason to suppose that the use of similar approximations will lead to substantial over hedging of spread option positions.

## REFERENCES

- C. Alexander and A. Scourse. Bivariate normal mixture spread option valuation. *Quantitative Finance*, 4(6):637–648, 2004.
- M. Broadie and J. Detemple. The valuation of american options on multiple assets. *Mathematical Finance*, 7(3):241–286, 1997.
- R. Carmona and V. Durrleman. Pricing and hedging spread options in a log-normal model. Technical report, Technical Report, 2003a.
- R. Carmona and V. Durrleman. Pricing and hedging spread options. *Siam Rev*, 45(4):627–685, 2003b.
- R. Carmona and V. Durrleman. Generalizing the black-scholes formula to multivariate contingent claims. *Journal Of Computational Finance*, 9(2):43, 2005.
- P. Carr and D. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2(4):61–73, 1999.
- P. Carr, R.A. Jarrow, and R. Myneni. *Alternative Characterizations of American Put Options*. Cornell University, Johnson Graduate School of Management, 1989.
- M. Dempster and S.S.G. Hong. Spread option valuation and the fast fourier transform. *Mathematical Finance, Bachelier Congress, Geman, H., Madan, D., Pliska, S.R. and Vorst, T., Eds., Springer Verlag, Berlin*, 1:203–220, 2000.
- J. Detemple. *American-style Derivatives: Valuation and Computation*. Chapman & Hall/CRC, 2005.
- V. Durrleman. Implied correlation and spread options. Technical report, Technical Report, Department Of Operations, 2001.
- Alexander Eydeland and Krzysztof Wolyniec. *Energy and power risk management: new developments in modeling, pricing and hedging*. John Wiley & Sons, 2003.
- S.D. Jacka. Optimal stopping and the american put. *Math. Finance*, 1(1):14, 1991.
- P. James. *Option theory*. Wiley New York, 2003.
- I.N. Kim. The analytic valuation of american options. *Review of Financial Studies*, 1990.
- E. Kirk. Correlation in energy markets. *Managing Energy Price Risk*, 1996.
- Minqiang Li, Shijie Deng, and Jieyun Zhou. Closed-form approximations for spread option prices and greeks. *SSRN eLibrary*, 2006.
- W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33(1):177–186, 1978.

- H.P. McKean. Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review*, 6(2):32–39, 1965.
- K. Ravindran. Low-fat spreads. *RISK*, 6(10):56–57, 1993.
- M. Rubinstein. One for another. *Risk*, 4(10):30–32, 1991.
- D. Shimko. Options on futures spreads: hedging, speculation and valuation. *Journal of Futures Markets*, 14(2):183–213, 1994.

## A APPENDIX: DERIVATION OF KIRK'S APPROXIMATION

In this appendix we derive the approximate pricing formula presented in Kirk [1996]. The derivation has not been documented in the literature, and neither were dividends included in the formula.

The payoff to a spread option is given by

$$[\omega(S_{1,T} - S_{2,T} - K)]^+ = (K + S_{2,T})[\omega(Z_T - 1)]^+$$

where  $Z_t = S_{1,t}/Y_t$  and  $Y_t = S_{2,t} + Ke^{-r(T-t)}$ . By Ito's lemma:

$$\begin{aligned} dZ_t &= \frac{\partial Z_t}{\partial S_{1,t}} dS_{1,t} + \frac{\partial Z_t}{\partial Y_t} dY_t + \frac{1}{2} \frac{\partial^2 Z_t}{\partial S_{1,t}^2} (dS_{1,t})^2 + \frac{1}{2} \frac{\partial^2 Z_t}{\partial Y_t^2} (dY_t)^2 + \frac{1}{2} \frac{\partial^2 Z_t}{\partial S_{1,t} \partial Y_t} (dS_{1,t} dY_t) \\ \frac{dZ_t}{Z_t} &= \left( \frac{dS_{1,t}}{S_{1,t}} - \frac{dY_t}{Y_t} + \left( \frac{dY_t}{Y_t} \right)^2 - \frac{dS_{1,t}}{S_{1,t}} \frac{dY_t}{Y_t} \right) \end{aligned} \quad (31)$$

We have  $dY_t = dS_{2,t} + Kre^{-r(T-t)} dt$  and for  $K \ll S_2$

$$\begin{aligned} \frac{dY_t}{Y_t} &= \frac{S_{2,t}}{Y_t} ((r - q_2) dt + \sigma_2 dW_{2,t}) \\ &= (\bar{r} - \bar{q}_2) dt + \bar{\sigma}_2 dW_{2,t} \end{aligned} \quad (32)$$

where

$$\bar{\sigma}_2 = \left( \frac{S_{2,t}}{Y_t} \right) \sigma_2, \quad \bar{r} = \left( \frac{S_{2,t}}{Y_t} \right) r, \quad \text{and} \quad \bar{q}_2 = \left( \frac{S_{2,t}}{Y_t} \right) q_2$$

are assumed to be constant. Hence (31) can be rewritten:

$$\frac{dZ_t}{Z_t} = (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\bar{\sigma}_2^2 - \sigma_1 \bar{\sigma}_2 \rho) dt + \sigma_1 dW_{1,t} - \bar{\sigma}_2 dW_{2,t}$$

Let  $W_{3,t}$  be a Brownian motion that is uncorrelated with  $W_{2,t}$  and such that

$$dW_{1,t} = \rho dW_{2,t} + \sqrt{1 - \rho^2} dW_{3,t}$$

Then,

$$\frac{dZ_t}{Z_t} = (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\bar{\sigma}_2^2 - \sigma_1 \bar{\sigma}_2 \rho) dt + (\rho \sigma_1 - \bar{\sigma}_2) dW_{2,t} + \sigma_1 \sqrt{1 - \rho^2} dW_{3,t}$$

Define  $dW_{2,t}^* = dW_{2,t} - \bar{\sigma}_2 dt$ . Using Girsanov's theorem, let  $\mathbb{P}$  be the new probability measure under which both  $W_{2,t}^*$  and  $W_{3,t}$  are martingales. The Radon-Nikodym derivative with respect to the risk-neutral probability  $\mathbb{Q}$  is then given by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{(-\frac{1}{2}\bar{\sigma}_2^2 T + \bar{\sigma}_2 W_{2,t})}$$

We now have

$$\begin{aligned} \frac{dZ_t}{Z_t} &= (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\sigma_1 \rho - \bar{\sigma}_2) dW_{2,t}^* + \sqrt{1 - \rho^2} \sigma_1 dW_{3,t} \\ &= (r - \bar{r} - (q_1 - \bar{q}_2)) dt + \sigma_t dW_t, \quad \text{say} \end{aligned}$$

The standard deviation of  $W_t$  is given by

$$\begin{aligned}\sigma &= \sqrt{(\sigma_1\rho - \bar{\sigma}_2)^2 + (1 - \rho^2)\sigma_1^2} \\ &= \sqrt{\sigma_1^2 + \sigma_2^2\left(\frac{S_{2,t}}{Y_t}\right)^2 - 2\rho\sigma_1\sigma_2\left(\frac{S_{2,t}}{Y_t}\right)}\end{aligned}\quad (33)$$

since  $W_{2,t}^*$  and  $W_{3,t}$  are independent Weiner processes.

Note that  $Z_t$  is (approximately) log-normal and is also observable in the market. Hence the spread option can be priced by treating it as a plain vanilla option defined on an observable asset whose price process is described by  $Z_t$  and with a strike  $K = 1$ . Therefore the price  $P_t$  at time  $t$  for an option on  $S_{1,t}$  and  $S_{2,t}$  with strike  $K$ , maturity  $T$  and payoff  $[\omega(S_1 - S_2 - K)]^+$  is given by:

$$P_t = \omega \left( S_{1,t} e^{-q_1(T-t)} \Phi(\omega d_1^*) - (K e^{-r(T-t)} + S_{2,t}) e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \Phi(\omega d_2^*) \right) \quad (34)$$

where  $\omega = 1$  for a call and  $\omega = -1$  for a put,

$$\begin{aligned}d_1^* &= \frac{\ln(Z_t) + (r - \bar{r} + \bar{q}_2 - q_1 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2^* &= d_1^* - \sigma\sqrt{T-t}\end{aligned}$$