



# Average Portfolio Insurance Strategies

Jacques Pézier and Johanna Scheller

ICMA Centre  
Henley Business School  
University of Reading

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ICMA Centre • The University of Reading  
Whiteknights • PO Box 242 • Reading RG6 6BA • UK  
Tel: +44 (0)1183 788239 • Fax: +44 (0)1189 314741  
Web: [www.icmacentre.rdg.ac.uk](http://www.icmacentre.rdg.ac.uk)  
Director: Professor John Board, Chair in Finance  
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## ABSTRACT

We design average portfolio insurance (API) strategies with an investment floor and a buffer that is a power of a geometric average of the underlying asset price. We prove that API strategies are optimal for investors with hyperbolic absolute risk aversion who become progressively more risk averse over time. During the averaging period, API strategies reduce the proportion of wealth allocated to the risky asset, which is the traditional life cycle investment recommendation. We compare the sensitivities of the fair price of equivalent payoffs generated by average and constant proportion portfolio insurance strategies and illustrate the performance of API strategies.

**JEL Classification Codes:** G11, G12, G13, G17

**Keywords:** Portfolio insurance, constant proportion portfolio insurance, average price Asian options, optimal payoff profile, utility theory, life cycle portfolio choice, power options.

### **Authors Details:**

Jacques Pézier  
Visiting Professor, ICMA Centre, Business School,  
Email: j.pezier@icmacentre.ac.uk

Johanna Scheller  
PhD candidate, ICMA Centre, Business School,  
Email: j.scheller@icmacentre.ac.uk

The University of Reading, PO Box 242  
Reading RG6 6BA, United Kingdom  
Tel: +44 (0)1183 786675 (ICMA Centre) Fax: +44 (0)1189 314741

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## 1. Introduction

Most investors are risk averse and require investment strategies which limit their exposure to risk. Merton (1971) and Brennan and Solanki (1981) show that the optimal payoff for an investor with an hyperbolic absolute risk aversion (HARA) utility in a world with a risk-free and a log-normally distributed risky asset consists of an investment floor plus a power of the risky asset price. Perold (1986) and Black and Jones (1987) introduce the constant proportion portfolio insurance (CPPI) strategy which replicates this optimal payoff. The CPPI is specified through two exogenous parameters: a floor and a multiplier. The investment floor splits the portfolio into a guaranteed part and a buffer, the excess over the floor. A constant multiple of the buffer is invested in the risky asset and the rest in the risk-free asset. The greater the floor and the lower the multiplier the lower the exposure to the risky asset and vice versa. The most common alternative portfolio insurance strategy to CPPI is the option based portfolio insurance strategy.

Portfolio insurance strategies suit long-term investors who save for their pensions. In theory, with a standard CPPI strategy in continuous time, the value of the portfolio is path independent; it depends solely on the price of the underlying risky asset and time. Volatile markets, as observed over last few years, make the final value highly uncertain. An average price (Asian) option decreases an investor's dependency on the final price of the underlying as its payoff is a function of the average price before maturity. The first study on pricing standard (calls and puts) Asian options is Kemna and Vorst (1990), who find that there is no analytic pricing formula for Asian options on arithmetic means as the continuous arithmetic averaging of log-normal distributions cannot be written analytically. They derive prices using Monte Carlo simulations and find that arithmetic Asian options are always cheaper than standard European options. Pricing options on a geometric average of log-normal prices on the other hand, yields simple analytical results. Kemna and Vorst (1990) and Turnbull and Wakeman (1991) evaluate geometric average price options; they are lower bounds for the prices of arithmetic average price options. A vast stream of literature follows on approximations for pricing options on arithmetic averages. We refer the reader to the following studies: Carverhill and Clewlow (1990) use fast Fourier transforms, Turnbull and Wakeman (1991) approximate the average by fitting integer moments, Geman and Yor (1993) apply a Laplace transform approach and Rogers and Shi (1995) derive a lower bound for the average asset price which comes very close to the true price.

We combine the concepts of average price options and portfolio insurance strategies by designing an average portfolio insurance (API) strategy. We define the payoff of an API strategy as a power of the geometric average of the underlying asset plus an investment floor. Such API strategies provide a capital guarantee and an upside potential which, unlike standard CPPI

strategies, depends on the price path of the underlying price. Investors do not risk losing a large fraction of their investment just before maturity as the increasing dependency on past prices decreases the volatility of the portfolio as it approaches maturity. The decreasing portfolio exposure to the risky asset is in line with classic life cycle investment strategies. Thus API strategies suit investors who become progressively more risk averse over time.

There are other extensions and alternatives to the standard CPPI strategies. El Karoui *et al.* (2005) and Attaoui and Lacoste (2010) design American type capital guarantees. Boulier and Kanniganti (1995) introduce variable floor CPPI strategies that adjust the floor should the exposure to the risky asset become too low or too high. Chen and Chang (2005) make the multiplier variable to adjust to changing market environments. Our extension of the CPPI spectrum suits investors who want to decrease their the exposure to risk over time. The literature has arguments for and against this life cycle investment recommendation. Bodie *et al.* (1992) stress that the optimal asset mix over a lifetime should be constant under simplistic assumptions as (namely, i.i.d. returns on investment are independent from labour income and consumption and under time-independent utilities). On the other hand, studies by Jagannathan and Kocherlakota (1996) and Cocco *et al.* (2005) argue that labour income or human capital should be introduced as an additional component in the risky asset mix that compensates for investment risks but decreases over time. Consequently, individuals should take more risky investments when they are young and move towards risk-free investments with increasing age when their capacity to earn decreases. Progressive habit formation leading to greater risk aversion is another argument for reducing investment risks over time.

The outline of the paper is as follows. Section 2 describes the assumptions we make on the financial market and defines the averaging process. We construct an API strategy that replicates a power payoff on a geometric average in Section 3 and derive the sensitivities of the fair price of this payoff in Section 4. In Section 5 we show the optimality of API strategies for HARA utility investors with risk tolerance decreasing uniformly over time. We illustrate the performance of several API strategies in Section 6 and conclude in Section 7.

## 2. The Market Model and the Geometric Average Process

We assume a Black–Scholes economy, that is, a complete financial market in which investors can freely trade in a risk-free asset and a risky asset; the risk-free asset grows at a constant rate  $r^f$  and the risky asset price follows a geometric Brownian process

$$S(t) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_S(t) \right), \quad (1)$$

where  $S(0)$  is the initial asset price, the constants  $\mu$  and  $\sigma$  denote the drift and the standard deviation, respectively, and  $W_S(t)$  is a Brownian process. There are no trading restrictions nor transaction costs. The standard CPPI strategy dynamically rebalances in continuous time a portfolio invested in the risk-free and the risky asset so as to maintain the exposure to the risky asset at a constant multiple of the buffer, the excess value of the portfolio above a floor that grows at the risk-free rate. The payoff can be found by applying Itô's Lemma to the strategy dynamics. Perold and Sharpe (1988) derive the CPPI payoff  $CPPI(t; F, m)$  for any  $t > 0$  as

$$\begin{aligned} CPPI(t; F, m) &= Fe^{r^f t} + B_P(t) \\ &= Fe^{r^f t} + (w(0) - F) \left( \frac{S(t)}{S(0)} \right)^m \exp \left( (1 - m) \left( r^f + \frac{1}{2} m \sigma^2 \right) t \right), \end{aligned} \quad (2)$$

where  $B_P(t)$  denotes the buffer,  $w(0)$  the initial wealth,  $F$  the initial investment floor and  $m$  the multiplier. Thus the payoff of a standard CPPI is a power function of the underlying. CPPI strategies are therefore the replication strategies of power options. As with standard options, the CPPI payoff is independent of the underlying asset price path. But, unlike standard options with payoffs defined at set maturities, CPPI strategies' power payoffs can be extended to any maturity; CPPI strategies are open ended. The multiplier controls the curvature of the strategy payoff: Multipliers greater than 1 give convex payoffs and multipliers smaller than 1 give concave payoffs (see Bertrand and Prigent, 2005). The choice of the floor affects only the volatility of the strategy but the multiplier influences all moments of the payoff distribution.

To construct an API strategy we define an average price and use it as the underlying index. We consider an averaging period from time  $t_a$  to maturity  $T$ , with  $0 \leq t_a \leq T$ . The two most commonly used types of averages are the arithmetic and the geometric averages. The arithmetic average sums up prices over the averaging period:

$$A(t_a, T) = \frac{1}{\tau} \int_{t_a}^T S(t) dt, \quad (3)$$

where  $\tau = T - t_a$ . The geometric average sums up the log-prices over the averaging period:

$$G(t_a, T) = \exp \left( \frac{1}{\tau} \int_{t_a}^T \ln(S(t)) dt \right). \quad (4)$$

The difference between Equation (3) and Equation (4) may be small but the geometric average is always lower than the arithmetic average for any volatile price (see Beckenbach and Bellman, 1961).

Kemna and Vorst (1990) and Turnbull and Wakeman (1991) derive the geometric average  $G_{t_a}$  of a geometric Brownian price given by Equation (1) over the period  $[t_a, T]$  as a function of

$S(t_a)$  and obtain

$$G_{t_a}(t_a, T) = S(t_a) \exp \left( \left( \mu - \frac{1}{2}\sigma^2 \right) \frac{1}{2}\tau + \sigma W \left( \frac{1}{3}\tau \right) \right), \quad (5)$$

where  $W(\tau)$  is a Brownian process correlated with  $W_S(\tau)$  but distinct from it. Compared to the underlying price process, the geometric average has half the drift and a third of the variance. Consequently, in a risk-neutral world with  $\mu = r^f$  its discounted value is not a martingale; the geometric average is not a tradable asset. To examine API strategies starting before the averaging period, we include the stock prices before  $t_a$  and derive the price of the geometric average at  $T$  as a function of  $S(0)$ , which we denote by  $G(t_a, T)$ .

**Proposition 2.1.** *The time  $T$  value of the geometric average process over interval  $[t_a, T]$  as a function of the initial price  $S(0)$ ,  $0 \leq t_a \leq T$  is*

$$G(t_a, T) = S(0) \exp \left( \left( \mu - \frac{1}{2}\sigma^2 \right) \left( T - \frac{1}{2}\tau \right) + \sigma W \left( T - \frac{2}{3}\tau \right) \right). \quad (6)$$

*Proof.* We start from the definition in Equation (4) of a geometric average on stock prices and obtain

$$\begin{aligned} G(t_a, T) &= \exp \left( \ln(S(t_a)) + \frac{1}{\tau} \int_{t_a}^T \ln(S(t)) dt \right) \\ &= \exp \left( \ln(S(0)) + \left( \mu - \frac{1}{2}\sigma^2 \right) t_a + \sigma W_S(t_a) + \left( \mu - \frac{1}{2}\sigma^2 \right) \frac{1}{2}\tau + \frac{1}{\tau} \int_{t_a}^T \sigma W_S(t) dt \right) \\ &= S(0) \exp \left( \underbrace{\left( \left( \mu - \frac{1}{2}\sigma^2 \right) \left( T - \frac{1}{2}\tau \right) + \sigma W_S(t_a) + \frac{1}{\tau} \int_{t_a}^T \sigma W_S(t) dt \right)}_X \right). \end{aligned}$$

The random term  $X$  in the exponent is the sum of normal distributions, it is therefore also normally distributed. Its expectation and variance are

$$\begin{aligned} \mathbb{E}[X] &= \left( \mu - \frac{1}{2}\sigma^2 \right) \left( T - \frac{1}{2}\tau \right) \quad \text{and} \\ \mathbb{V}[X] &= \sigma^2 \left( T - \frac{2}{3}\tau \right). \end{aligned}$$

Thus the closed form of the average price process is equal to

$$G(t_a, t) = \begin{cases} S(0) \exp \left( \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_S(t) \right) & \text{if } 0 \leq t \leq t_a \\ S(0) \exp \left( \left( \mu - \frac{1}{2}\sigma^2 \right) \left( t - \frac{1}{2}\tau \right) + \frac{\sigma}{\sqrt{3}} W \left( t - \frac{2}{3}\tau \right) \right) & \text{if } t_a < t \leq T. \end{cases}$$

□

### 3. Construction of an API Strategy

The standard CPPI strategy leads to the power payoff in Equation (2). Similarly, we define an API strategy as the dynamic asset allocation strategy in continuous time replicating the corresponding payoff based on the geometric average price over the time interval  $[t_a, T]$ , that is, at maturity  $T$ :

$$A(T; F, m, t_a, T) = F e^{r^f T} + (w(0) - F) k_{\text{API}} \left( \frac{G(t_a, T)}{S(0)} \right)^m. \quad (7)$$

The buffer is now a power function of the geometric average as in Equation (6) and  $k_{\text{API}}$  is a normalisation factor such that the risk-neutral price of the API payoff under the risk-neutral probability  $\mathbb{Q}$  is equal to initial wealth  $w(0)$ . With the filtration  $\mathcal{F}_0$  at  $t = 0$  this yields

$$A(0; F, m, t_a, T) = \mathbb{E}_{\mathbb{Q}} \left[ A(T; F, m, t_a, T) e^{-r^f T} | \mathcal{F}_0 \right] = w(0)$$

and hence

$$\begin{aligned} k_{\text{API}} &= \mathbb{E}_{\mathbb{Q}} \left[ \left( \frac{G(t_a, T)}{S(0)} \right)^m e^{-r^f T} | \mathcal{F}_0 \right]^{-1} \\ &= \exp \left( -m \left( r^f - \frac{1}{2} \sigma^2 \right) \left( T - \frac{1}{2} \tau \right) - \frac{1}{2} m^2 \sigma^2 \left( T - \frac{2}{3} \tau \right) + r^f T \right). \end{aligned} \quad (8)$$

When we take the expectation of the average we have to fix the maturity of the strategy and lose the open-endedness of standard CPPI strategies. To replicate the payoff in Equation (7) we determine the fair price<sup>1</sup>  $A(t)$  at time  $t$  for  $0 \leq t \leq T$ . If  $0 \leq t \leq t_a$  the fair price depends only on  $S(t)$  since the averaging has not started yet and the fair price is path independent. As soon as we enter the averaging period, i.e.  $t_a < t \leq T$ , the fair price depends also on the realised average between  $t_a$  and  $t$ : In  $[0, t_a]$   $\mathcal{F}_t$  contains the knowledge of  $S(t)$  only and in  $(t_a, T]$   $\mathcal{F}_t$  contains the knowledge of  $S(t)$  and  $G(t_a, t)$ . Hence we consider the two time intervals  $[0, t_a]$  and  $(t_a, T]$  separately to price the API payoff.

**Theorem 3.1.** *When  $0 \leq t \leq t_a$  the risk-neutral price of the API strategy with a payoff as*

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<sup>1</sup>To simplify notation we refer to the fair API price as  $A(t)$ .

defined in Equation (7) is

$$\begin{aligned}
A(t) &= F e^{r^f t} + B_{API}(t) \\
&= F e^{r^f t} + (w(0) - F) k_{API} \left( \frac{S(t)}{S(0)} \right)^m \\
&\quad \exp \left( m \left( r^f - \frac{1}{2} \sigma^2 \right) \left( T - t - \frac{1}{2} \tau \right) + \frac{1}{2} m^2 \sigma^2 \left( T - t - \frac{2}{3} \tau \right) - r^f (T - t) \right). \quad (9)
\end{aligned}$$

When  $t_a < t \leq T$  the risk-neutral price of the API payoff is

$$\begin{aligned}
A(t) &= F e^{r^f t} + B_{API}(t) \\
&= F e^{r^f t} + (w(0) - F) k_{API} \left( G(t_a, t)^{1-\alpha(t)} \frac{S(t)^{\alpha(t)}}{S(0)} \right)^m \\
&\quad \exp \left( \alpha m \left( r^f - \frac{1}{2} \sigma^2 \right) \frac{T-t}{2} + \frac{1}{2} \alpha^2 m^2 \sigma^2 \frac{T-t}{3} - r^f (T-t) \right). \quad (10)
\end{aligned}$$

where  $\alpha(t) = \frac{T-t}{\tau}$ .

*Proof.* We distinguish between Case 1 and 2 as the time before and during the averaging period:

- Case 1:  $0 \leq t \leq t_a$

Using Equation (5) the risk-neutral expectation is

$$A(t) = F e^{r^f t} + (w(0) - F) k_{API} \mathbb{E}_Q[G(t_a, T)^m e^{-r^f(T-t)} | S(t)],$$

which gives Equation (9) when evaluating the expectation.

- Case 2:  $t_a < t \leq T$

As we know the price process path until  $t$  we split the geometric average  $G(t_a, T)$  into the known and unknown part and weigh them respectively:

$$G(t_a, T) = G(t_a, t)^{1-\alpha(t)} G(t, T)^{\alpha(t)}.$$

With Equation (12) we obtain

$$A(t) = F e^{r^f t} + (1 - F) k_{API} G(t_a, t)^{m(1-\alpha(t))} \mathbb{E}_Q[G(t, T)^{m\alpha(t)} e^{-r^f(T-t)} | S(t)].$$

Evaluating the expectation of the weighed  $G(t, T)$  gives the risk-neutral price in Equation (10). The second term in the sum in Equation (10) is the buffer  $B_{API}$ .

□

Taking into account  $k_{\text{API}}$ , at  $0 \leq t \leq t_a$  the API strategy is a standard CPPI given by Equation (2) and when  $t_a < t \leq T$  we obtain the risk-neutral price

$$A(t) = F e^{r^f t} + (w(0) - F) \left( G(t_a, t)^{1-\alpha(t)} \frac{S(t)^{\alpha(t)}}{S(0)} \right)^m \exp \left( \frac{1}{2} m \left( r^f - \frac{1}{2} \sigma^2 \right) (\alpha(t)^2 \tau - (2T - \tau)) + \frac{1}{6} m^2 \sigma^2 (\alpha(t)^3 \tau - (3T - 2\tau)) + r^f t \right).$$

In the special case where averaging starts at  $t_a = 0$  and therefore  $\tau = T$ , which we call a Full API, we obtain

$$A(t) = F e^{r^f t} + (w(0) - F) \left[ G(0, t)^{1-\alpha(t)} \frac{S(t)^{\alpha(t)}}{S(0)} \right]^m \exp \left( \frac{1}{2} m \left( r^f - \frac{1}{2} \sigma^2 \right) ((\alpha(t)^2 - 1)T) + \frac{1}{6} m^2 \sigma^2 ((\alpha(t)^3 - 1)T) + r^f t \right)$$

for all  $t \in [0, T]$ .

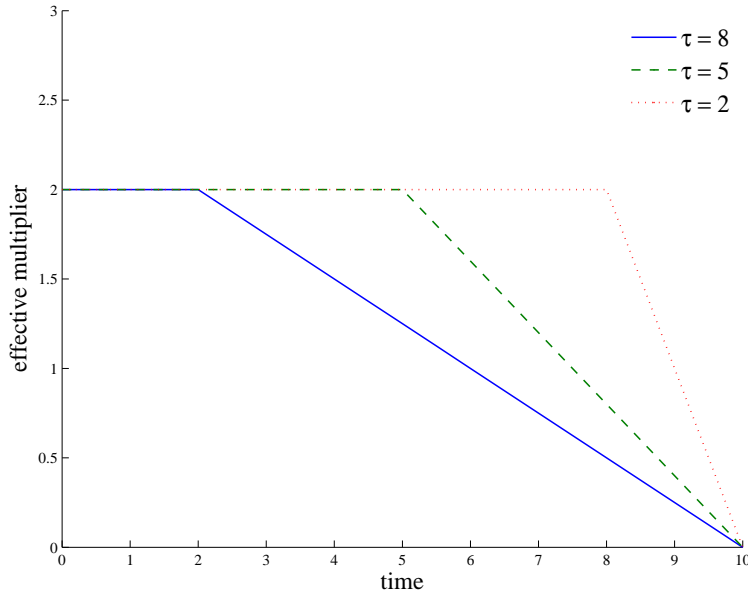


FIGURE 1: *Effective API multiplier as a function of time for  $m = 2$ ,  $T = 10$  and three averaging periods  $\tau$*

Compared to the constant CPPI multiplier  $m$ , we see that the effective API multiplier on  $S(t)$  is  $\alpha(t)m$  and decreases linearly from  $m$  to 0 during the period of averaging. Thus API investors reduce their exposure to the risky asset over time and thereby the uncertainty in the final value of their portfolio. This is in line with the traditional life cycle investment recommendation that investors should reduce progressively their exposure to risky assets as a long-term strategy. In Figure 1 we show the effective API multiplier for three values of the averaging period  $\tau$ . The

multiplier starts decreasing when the averaging process begins. It decreases slowly for long periods of averaging and fast for shorter periods to reach zero at maturity. The API price has the same curvature properties as the standard CPPI payoff: The payoff profile is convex if  $\alpha(t)m > 1$ , concave if  $\alpha(t)m < 1$  and linear if  $\alpha(t)m = 1$ .

#### 4. Sensitivities

We compare the sensitivities of fair prices of power payoffs, or power options, as we would for standard options, that is, we consider fixed equivalent CPPI and API power options and compare the sensitivities of the fair prices of these options to changes in the price and the price volatility of a common underlying risky asset. We take as equivalent payoffs that have the same maturity, floor and initial multiplier and that are calibrated to have the same initial fair price  $w(0)$ .<sup>2</sup> For the API option the fair price is given by Equation (9) and Equation (10). The equivalent CPPI option is given by Equation (2) when  $t = T$ , that is

$$P(T; F, m, T) = Fe^{r^f T} + (w(0) - F) k_p \left( \frac{S(T)}{S(0)} \right)^m, \quad (11)$$

where  $k_p$  is a constant. Setting the discounted payoff equal to initial wealth we obtain

$$\begin{aligned} k_p &= \mathbb{E}_{\mathbb{Q}} \left( \left( \frac{S(T)}{S(0)} \right)^m e^{-r^f T} \middle| \mathcal{F}_0 \right)^{-1} \\ &= \exp \left( (1 - m) \left( r^f + \frac{1}{2} m \sigma^2 \right) T \right). \end{aligned} \quad (12)$$

We now calculate the fair price of the CPPI option in Equation (11) at any time before maturity.

**Proposition 4.1.** *The risk-neutral price  $P(t; F, m, T)$  of the CPPI option with the payoff in Equation (11) at time  $t \in [0, T]$  is*

$$\begin{aligned} P(t; F, m, T) &= Fe^{r^f T} + B_p(T) \\ &= Fe^{r^f t} + (w(0) - F) k_p \left( \frac{S(t)}{S(0)} \right)^m \exp \left( (m - 1) \left( r^f + \frac{1}{2} m \sigma^2 \right) (T - t) \right) \end{aligned} \quad (13)$$

*Proof.* The risk-neutral price of the CPPI option at time  $t$  is equal to

$$P(t; F, m, T) = Fe^{r^f t} + (w(0) - F) k_p \frac{1}{S(0)^m} \mathbb{E}_{\mathbb{Q}}[S(T)^m e^{-r^f(T-t)} | S(t)]$$

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<sup>2</sup>Bertrand and Prigent (2005) examine the sensitivities of the payoff  $CPPI(t; F, m)$  to changes in the underlying price and changes in its volatility rather than the sensitivities of the fair price of a fixed option payoff.

which gives Equation (13) since

$$\mathbb{E}_{\mathbb{Q}}[S(T)^m e^{-r^f(T-t)} | S(t)] = S(t)^m \exp\left((m-1)\left(r^f + \frac{1}{2}m\sigma^2\right)(T-t)\right).$$

□

If we take  $k_P$  into account we recover the CPPI payoff given by Equation (2). For simplicity we shall refer to the fair price of the CPPI option as  $P(t)$ . We can now compare the sensitivities of the risk-neutral price of the CPPI option in Equation (2) to the sensitivities of the risk-neutral price of the API option in Equations (9) and (10).

The sensitivity of the CPPI option to changes in the underlying asset, the delta, of the power option is

$$\Delta_P(t) = \frac{\partial P(t)}{\partial S(t)} = m \frac{1}{S(t)} B_P(t).$$

The delta of the API option differs for times before and during the averaging and we obtain

$$\Delta_{\text{API}}(t) = \frac{\partial \text{API}(t)}{\partial S(t)} = \begin{cases} m \frac{1}{S(t)} B_{\text{API}}(t) & \text{if } 0 \leq t \leq t_a \\ \alpha(t) m \frac{1}{S(t)} B_{\text{API}}(t) & \text{if } t_a < t \leq T. \end{cases}$$

The deltas of both options are always positive, meaning that the greater the price of the underlying the greater the prices of these options. It is theoretically possible in a Black–Scholes economy to replicate the payoffs of the CPPI option in Equation (13) and the API option as in Equations (9) – (10) by maintaining in continuous time an exposure  $\Delta_P S(t)$  and  $\Delta_{\text{API}} S(t)$  to the risky asset, respectively. For API the delta decreases during the averaging period until it reaches 0 at maturity as the payoff becomes more heavily dependent on the average and less sensitive to changes in the underlying. We need the delta of the API strategy relative to the risky asset price to replicate the payoff as the geometric average is not a tradable asset.

The sensitivity of the delta to changes in the underlying, the gamma, of the CPPI option is

$$\Gamma_P(t) = \frac{\partial \Delta_P(t)}{\partial S(t)} = m(m-1) \frac{1}{S(t)^2} B_P(t),$$

whereas the gamma of the API option is

$$\Gamma_{\text{API}}(t) = \frac{\partial \Delta_{\text{API}}(t)}{\partial S(t)} = \begin{cases} m(m-1) \frac{1}{S(t)^2} B_{\text{API}}(t) & \text{if } 0 \leq t \leq t_a \\ \alpha(t)m (\alpha(t) m - 1) \frac{1}{S(t)^2} B_{\text{API}}(t) & \text{if } t_a < t \leq T. \end{cases}$$

For  $m > 1$  the gamma of the API option turns from positive to negative during the averaging period.

Likewise the sensitivity to changes in the underlying's volatility, or the vega of the CPPI option is<sup>3</sup>

$$v_P(t) = \frac{\partial P}{\partial \sigma} = \sigma m(m-1)(T-t)B_P(t),$$

whereas the vega of the API option is

$$v_{API}(t) = \frac{\partial API}{\partial \sigma} = \begin{cases} \sigma m [(m-1)(T-t) + (\frac{1}{2} - \frac{2}{3}m)\tau] B_{API}(t) & \text{if } 0 \leq t \leq t_a \\ \alpha \sigma m (\alpha m^{\frac{T-t}{3}} - \frac{T-t}{2}) B_{API}(t) & \text{if } t_a < t \leq T. \end{cases}$$

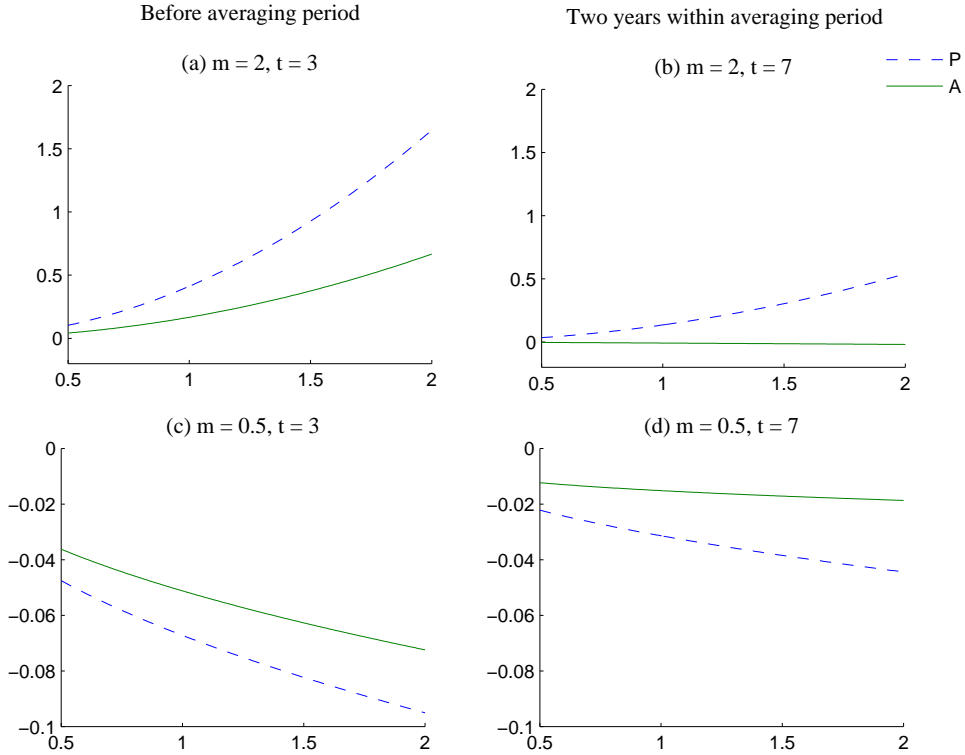


FIGURE 2: Vegas of CPPI option  $P$  and API option  $A$  at time  $t$  as a function of the risky asset price  $S$  for  $T = 10$ ,  $\tau = 5$ ,  $\sigma = 18\%$ ,  $r^f = 3.5\%$ ,  $F = 0.8$  and  $G(5, 7) = 1$  (the horizontal axes show the value of the risky asset  $S(t)$  at  $t$  from 0.5 to 2 and the vertical axes the vegas)

Figure 2 illustrates the vegas of the CPPI option and the API option.<sup>4</sup> The vega of the CPPI option is positive if  $m > 1$  and negative if  $m < 1$ . The vega of the API option for  $t \leq t_a$  is

<sup>3</sup>The constants  $k_P$  and  $k_{API}$  do not impact the vegas as they are set at strategy initiation.

<sup>4</sup>The market parameters used in Figure 2 are approximately the mean and the standard deviation of the S&P500 from 1990 to 2010.

positive if

$$m > \frac{T - t - \frac{1}{2}\tau}{T - t - \frac{2}{3}\tau}.$$

During the averaging period the vega of the API depends on the multiplier relative to  $T$ ,  $t$  and  $\tau$ . We find that  $v_{\text{API}}$  is positive during the time of averaging for  $m > \frac{3}{2\alpha}$  (compare Figures 2(a) and (b) with (c) and (d)). For the chosen set of parameters the vega is positive at  $t = 3$  if  $m > 1.23$  and at  $t = 7$  for multipliers greater than 2.50.

## 5 Optimality of the API Option

The traditional argument to justify life cycle investment strategies is to consider human capital as an additional asset which decreases over time. It is assumed that earnings generated by human capital are negatively correlated with investment performance, that is, if investment performance is poor, human capital can compensate by generating more income and vice versa, good investment performance can provide more leisure time. Alternatively or in addition one may assume an increasingly risk averse utility function over time. We use the following specification of a HARA utility:

$$u(\tilde{w}) = \text{sgn}(\eta - 1) \left( 1 + \frac{\eta(t)}{\lambda(t)w(0)} (\tilde{w} - w(0)) \right)^{1 - \frac{1}{\eta(t)}},$$

where  $\tilde{w}$  denotes the present value of future wealth and  $w(0)$  is the investor's initial wealth. The parameters specifying the utility function are  $\lambda(t)$ , a function of the current local coefficient of absolute risk tolerance  $\lambda_0$ , itself defined as a proportion of initial wealth, and  $\eta(t)$ , the sensitivity of risk tolerance to changes in wealth. The absolute risk tolerance (ART), which is the inverse of Pratt's absolute risk aversion, is

$$\text{ART} = -\frac{u'}{u''} = \lambda(t)w(0) + \eta(t)(\tilde{w} - w(0)).$$

The ART changes with wealth at the rate  $\eta(t)$ , which is likely to be positive for most investors. With constant parameters  $\lambda$  and  $\eta$ , Brennan and Solanki (1981) find that the optimal investment payoff when the underlying price is log-normally distributed with excess log-returns  $r \sim N(\mu - \frac{1}{2}\sigma^2, \sigma^2)$  is

$$w(T)e^{-r^f T} = F^* + (1 - F^*) \exp \left( m^* r T + \frac{1}{2} (1 - m^*) m^* \sigma^2 T \right), \quad (14)$$

where  $m^* = \eta(t) \frac{\mu}{\sigma^2}$  is the optimal multiplier and  $F^* = 1 - \frac{\lambda(t)}{\eta(t)}$  is the optimal floor. This payoff is the CPPI payoff in Equation (2).<sup>5</sup>

Consider now that the risk aversion of investors increases over time. Specifically, assume that their ART decreases at the rate  $\alpha(t)$  by decreasing both parameters  $\lambda(t)$  and  $\eta(t)$  at the same rate, that is set  $\lambda(t) = \alpha(t)\lambda_0$  and  $\eta(t) = \alpha(t)\eta_0$  with initial values  $\lambda_0$  and  $\eta_0$  at  $t = 0$ .

Then the optimal strategy parameters  $F^*$  and  $m^*$  change accordingly. From (14) we know that the optimal multiplier is a function of  $\eta(t)$  and hence changes with time:

$$m_{\text{API}}^*(t) = \eta(t) \frac{\mu}{\sigma^2} = \alpha(t)m^*.$$

But as  $\eta(t)$  and  $\lambda(t)$  change at the same rate  $\alpha(t)$  their ratio  $\frac{\lambda(t)}{\eta(t)}$  stays constant and consequently the investors' optimal floor remains constant at

$$F_{\text{API}}^* = 1 - \frac{\lambda_0}{\eta_0}.$$

Comparing the optimal strategy parameters of these investors with increasing risk aversion over time with the API strategy in Equation (10) shows a perfect match: The API payoff is optimal for HARA investors who change their utility function by decreasing  $\lambda(t)$  and  $\eta(t)$  to zero at the same uniform rate over time over the averaging period.

We show the effect of the changing  $\lambda(t)$  and  $\eta(t)$  on the ART at various times as a function of wealth in Figure 3. We consider a maturity of  $T = 10$  and an averaging period of  $\tau = 5$ . The ART is linear in wealth which is the case only for HARA utilities (see Gollier, 2001). The ART function has its root at  $F^* - w(0)$ . When the utility parameters change over time the slope of the ART function decreases, but the floor stays at its initial value.

Decreasing  $\eta(t)$  faster than  $\lambda(t)$  would increase risk tolerance for low levels of wealth and result in a lowering of the floor, which is always possible. On the other hand, decreasing  $\lambda(t)$  faster than  $\eta(t)$  would necessitate raising the floor of the optimal strategy faster than at the risk-free rate and could eventually force the entire portfolio into the risk-free asset.

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<sup>5</sup>A usual approach to measure the overall lifetime utility is to assume that it is the sum of the marginal utilities of consumption (see Gollier, 2001). Assuming log-normally distributed prices and constant consumption Merton (1971) also finds that for investors who have constant additive HARA type utilities the optimal payoff function is a power of the underlying plus a floor.

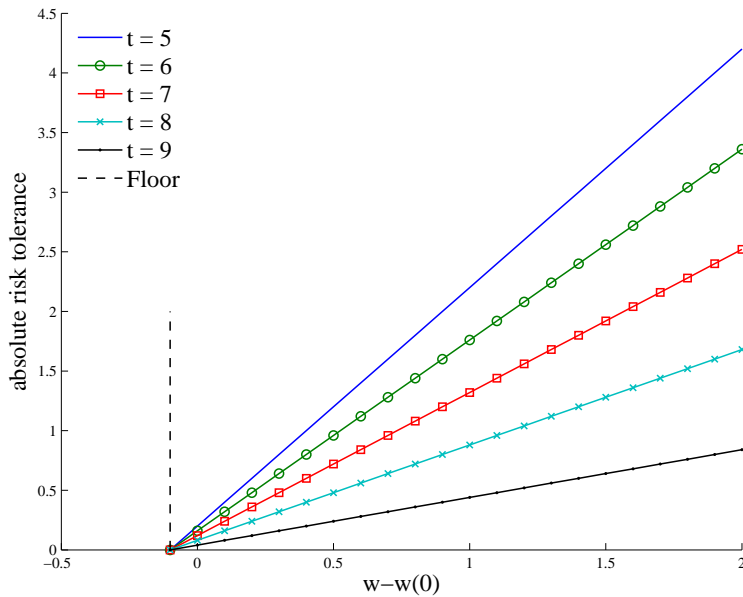


FIGURE 3: *ART as a function of excess wealth for  $\lambda_0 = 0.2$ ,  $\eta_0 = 2$ ,  $T = 10$  and  $\tau = 5$*

## 6. Performance Comparison of API Strategies

API strategies reduce progressively the exposure of the investor to the risky asset and therefore serve investors who, for whatever reason, want to follow the classic life cycle investment recommendation. To illustrate the effects of averaging and to show the differences between various API strategies, we simulate one path of the risky asset price following a geometric Brownian process with  $\mu = 5\%$  and  $\sigma = 18\%$  and plot the discounted paths of API strategies for various parameter combinations together with the discounted risky asset price in Figure 4. We use a risk-free rate of  $r^f = 3.5\%$ . The standard CPPI strategy has only two strategy parameters, the floor and the multiplier, but an API investor additionally needs to specify the start of the averaging period. We show three different choices of averaging periods:  $\tau = 8$ , 5 and 2 with an initial fund value of 1 and a time horizon of 10 years. Figure 4(a) shows the strategy with the greatest volatility as the floor is low and the multiplier high. In the first 2 years the portfolio values move like the underlying until investors with the longest averaging period  $\tau = 8$  start averaging, which reduces their portfolio volatility. None of the strategies dominates the others; the path of the underlying price determines which strategy has the highest final fund value. In this instance, as the underlying price decreases at the start of the averaging periods, the strategies with the longest averaging periods happen to outperform the strategies with the shorter averaging periods.

Comparing Figures 4(a)–4(d), we see that when the volatility of the fund value is high the multiplier is high, the floor is low, or both. The strategies in Figures 4(c) and 4(d), which have a low multiplier have the lowest volatility and the fund grows at about the risk-free rate.

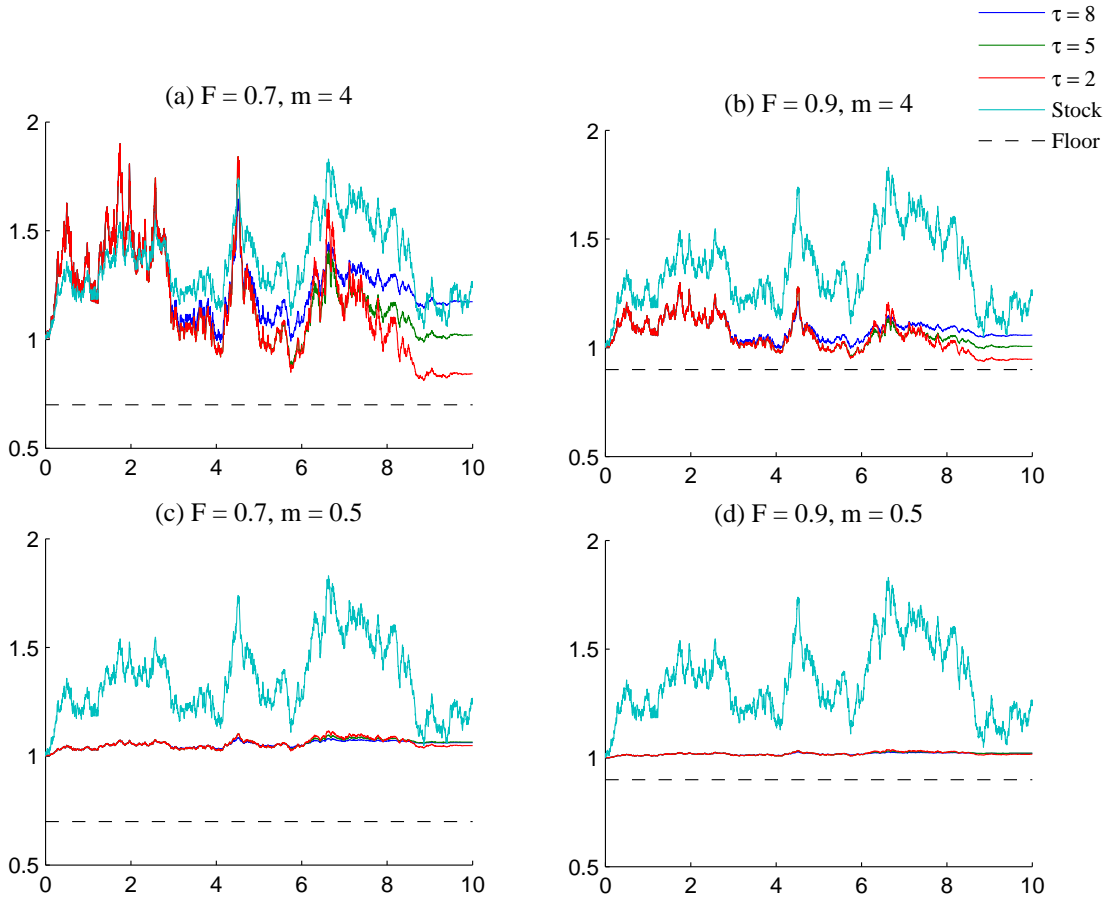


FIGURE 4: *Discounted fund value paths of API strategies for  $T=10$  and several floors and multipliers with an initial fund value of 1 (the horizontal axes show time from 0 to 10 years and the vertical axes the fund value)*

Naturally, the API strategies with the highest volatility have also the greatest upside potential.

Figure 5 shows the densities of the excess fund value of API strategies with various averaging periods. The time horizon is 1 year and the initial fund value is 1; we set the floor equal to 0.8 and the multiplier to 2. First we see that a Full API strategy with  $\tau = 10$  has the lowest volatility, skewness and kurtosis and therefore the highest peak. The other extreme is the standard CPPI strategy with  $\tau = 0$  which has the lowest peak. The densities of the strategies in Figure 5 approach the Full API if the averaging period is long and approach the standard CPPI if it is short.

## 7. Conclusion

We have defined average portfolio insurance payoffs that depend on the choice of a floor and a buffer that is a power of a geometric average of an underlying asset price. The investor can

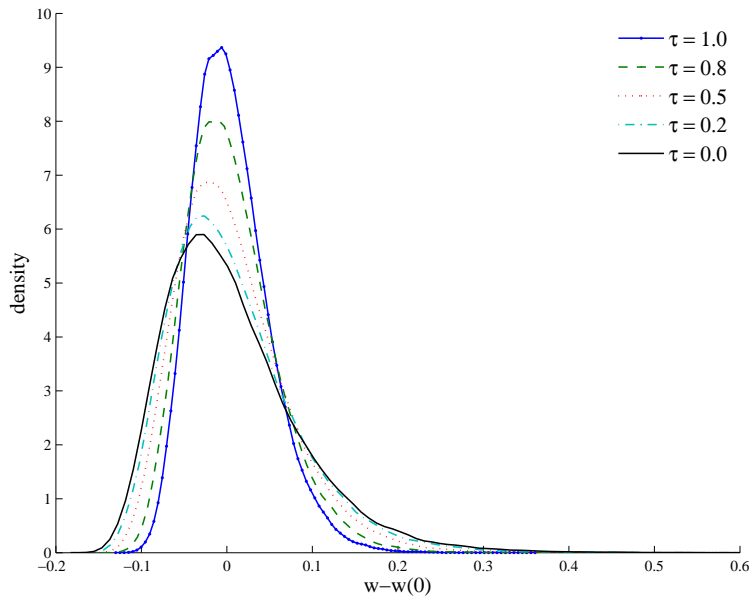


FIGURE 5: *Excess fund value densities of API strategies for  $T=1$  and several averaging periods for  $F=0.8$  and  $m=2$*

choose the averaging period to be less than or equal to the investment maturity. Whereas a CPPI strategy maintains the exposure to the risky asset at a constant multiple of the buffer, the multiplier in an API strategy decreases linearly with time, reducing the share of the portfolio invested in the risky asset to zero at maturity.

These characteristics make API strategies suitable for long-term investors who save for their pension. A common recommendation known as the life cycle investment strategy is to move progressively out of risky investments when an individual approaches retirement. The life cycle investment approach is justified both by the investor's habit forming (increasing risk aversion) and his decreasing capacity over time to generate labour income to compensate for poor investment performance. We prove that an API strategy is the optimal implementation of a life cycle investment for any investor with a HARA utility function provided that both his local coefficient of absolute risk tolerance and his sensitivity of risk tolerance to wealth decrease with time at the same rate. Together, the investor's risk tolerance parameters and the market parameters determine the optimal floor and multiplier of the API strategy.

API payoffs need to be defined for a certain maturity. Financial intermediaries could offer them to long-term investors in the same way as they offer other traditional and exotic structured products. Alternatively, long-term investors may choose to replicate these payoffs by implementing a dynamic investment strategy. To help manufacture API payoffs and assess the risk of a portfolio containing such structured products, we examine the sensitivities of their fair prices to changes in the underlying asset price and its volatility. We compare the API

sensitivities to those of a fixed-maturity CPPI with the same maturity and same initial value. The vega of an API payoff, before and during the averaging period, is lower than the vega of the equivalent CPPI payoff with same floor, initial multiplier, maturity and initial fair price. This proves that the API strategy does not only lower the portfolio volatility by taking the average but it also reduces the sensitivity of the strategy to changes in volatility. Finally, we use Monte Carlo simulations to illustrate how the choice of floor and multiplier influence the performance of an API strategy.

Thus, API strategies offer an optimal implementation of a life cycle investment strategy for an investor with ART decreasing linearly to zero over time. API strategies could be extended to suit investors with risk attitudes varying over time in more general ways.

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