A Comprehensive Evaluation
of Portfolio Insurance Strategies

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ABSTRACT

We present a comprehensive framework for comparing the merits of alternative portfolio insurance strategies in realistic contexts. Our findings add generality to previous results comparing option based and constant proportionality portfolio insurance strategies (OBPI and CPPI). The optimal OBPI and CPPI payoffs are determined by maximising expected utilities, with various degrees of risk sensitivity and over several investment horizons, using a general, two-parameter HARA utility. We consider two cases: either defined payoffs are purchased at fair prices or, as is typical in the implementation of portfolio insurance strategies, replicated discretely. The price dynamics of risky assets are modelled with either a geometric Brownian process or a time-changed geometric Brownian. Our results confirm the superiority of CPPI over OBPI in all cases. The effects of discrete replication and discontinuous price processes are examined by simulation and compared to the purchase at fair price of the theoretically optimal CPPI payoff when the underlying process is geometric Brownian.

JEL Classification Codes: G11, G17

Keywords: Capital guaranteed products, constant proportionality portfolio insurance, option based portfolio insurance, jump processes, time-changed Brownian motion, dynamic replication, utility theory.

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1 Introduction

Portfolio insurance strategies are designed to limit downside risk by insuring a predefined floor whilst allowing participation in the upside of a risky asset. Their popularity is increasing amongst investors that seek insurance not only against abrupt falls in the market, such as the crash in equities after the default of Lehman Brothers, but also general downturns such as the collapse of the dot.com bubble in the early 2000s.

Two popular strategies are option based portfolio insurance (OBPI) and constant proportion portfolio insurance (CPPI). OBPI strategies, that protect an investment with a put option on the risky asset, were first discussed by Leland and Rubinstein (1976). Alternatively one can secure a floor with an investment in a risk-free asset (a bond or a savings account) and the purchase of a call option on the risky asset.\(^1\) At maturity the portfolio value will always equal or exceed the current value of the risk-free asset, assuming it is provided by a creditworthy underwriter. The theoretical payoff at the maturity, \(T\), of the strategy is:

\[
C_{\text{OBPI}}(S^c, T) = F \exp(r^f T) + N \max(S^c(T) - K, 0),
\]

where \(S^c\) is the price of the risky asset at \(T\), \(F\) is the initial investment in the risk-free asset, \(r^f\) denotes the continuous risk-free rate of return and \(N\) is the number of call options bought at strike \(K\). OBPI is a static method if the call option can be bought, but in practice the call often needs to be replicated using a dynamic, discretely monitored investment strategy.

The CPPI approach was introduced by Perold (1986) and Black and Jones (1987) and further developed by Perold and Sharpe (1988). This strategy ensures a predefined floor by dynamically rebalancing allocations between a risky asset and a risk-free asset. A constant proportion or multiplier, \(m\), of the excess value of the investment above the floor (the buffer) is allocated to the risky asset, the rest is invested risk-free. The floor and the multiplier are exogenous variables to the model and are determined by the investor’s risk attitude and the investor’s views on the evolution of the risky asset. The lower the floor and the higher the multiplier, the greater the allocation to the risky asset. The investor then has a higher upside potential but the floor is reached more quickly if the risky asset price falls.

At the continuous time limit, assuming that the price of the risky asset follows a geometric Brownian diffusion, denoted \(S^c\):

\[
S^c(t) = S^c(0) \exp\left(\left(\mu - r^f - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)
\]

\(^1\)In the literature option based strategies are often modelled with portfolio and put options but, for ease of comparison with CPPI strategies, we use the bond plus call options in this paper.
with \( W(t) \) a Brownian motion, the CPPI method yields a path-independent power payoff at any investment horizon \( T \), given by

\[
C_{\text{CPPI}}(S^c, T) = F \exp(r^f T) + (w(0) - F) \left( \frac{S^c(T)}{S(0)} \right)^m \exp \left( \left(1 - m \right) \left( r^f + \frac{1}{2} m \sigma^2 \right) T \right),
\]

where \( w(0) \) denotes the investor’s initial wealth, \( S(0) \) denotes the initial portfolio price, \( \sigma \) is the constant diffusion coefficient and \( S^c(T) \) is the price of the risky portfolio at \( T \).\(^2\) Depending on the choice of multiplier the payoff may be concave \((0 < m < 1)\), linear \((m = 1)\) or convex \((m > 1)\). It seems intuitive that risk averse investors should value downside protection and convex CPPI payoffs, but that is not necessarily the case. We find conditions under which the optimal CPPI payoff for risk averse investors is concave.

As OBPI and CPPI strategies offer alternative protected payoffs, it is natural to examine under what circumstances an investor should prefer one type of protection to the other. Zhu and Kavee (1988) use Monte Carlo simulation to compare various sample statistics of replicated OBPI and CPPI payoffs. El Karoui et al. (2005) investigate which strategy over a finite horizon maximises a utility criterion and prove the optimality of OBPI strategies when a position in the risky portfolio requires a given level of guarantee. Bertrand and Prigent (2005), Annaert et al. (2009), and Zagst and Kraus (2011) compare OBPI and CPPI using stochastic dominance criteria. Bertrand and Prigent (2005) assume the geometric Brownian case and conclude that there is no evidence of strong or weak stochastic dominance between the two strategies, but there may be dominance of either strategy in a mean-variance sense, depending on the value of the CPPI multiplier. Zagst and Kraus (2011) extend this analysis to considering second and third order stochastic dominance, deriving conditions for the strategy parameters and market parametrisations such that CPPI stochastically dominates OBPI to the second order at maturity. Annaert et al. (2009) do not make assumptions on the dynamics of the risky asset and also find no stochastic dominance between OBPI and CPPI. Furthermore, they consider a broader range of performance measures without finding one strategy to outperform the other. Bertrand and Prigent (2011) show the dominance of CPPI strategies under Kappa performance measures by comparing theoretical payoffs.\(^3\)

When the risky asset price follows a geometric Brownian diffusion the portfolio value can, theoretically, never reach the floor, but in reality markets experience jumps. Benninga and Blume (1985) show that portfolio insurance strategies are still desired by investors in incomplete markets. In the presence of market jumps Bertrand and Prigent (2002) study how extreme

\(^2\)See Perold and Sharpe (1988).

\(^3\)Further studies on portfolio insurance strategies have been conducted by e.g. Brennan and Schwartz (1988), Black and Rouhani (1989), Black and Perold (1992), Bookstaber and Langsam (2000), Cesari and Cremonini (2003), and Bertrand and Prigent (2003).
moves in asset returns may impact portfolio insurance and Cont and Tankov (2009) examine the gap risk - the risk of falling below the floor - and derive the gap loss distribution and various associated risk measures in the context of a jump-diffusion model. Zhu and Kavee (1988) and Bertrand and Prigent (2011) compare OBPI and CPPI strategies with an underlying following a compound Poisson process.

Our analysis extends previous research as follows. We do not consider stochastic dominance but instead use a certainty equivalent return \((CER)\) as the satisfaction index; this is implemented with a general two-parameter HARA utility function, which encompasses most common types of utility functions, thus representing investors with very diverse risk preferences. Second, we argue that comparing a static OBPI strategy with a fixed payoff function to a dynamic CPPI strategy with replication costs and errors would be unfair. The CPPI payoff profile \((3)\) under continuous replication and a geometric Brownian price has a fair price that can be calculated as for standard options. We can therefore compare the two defined payoffs based on their fair prices in a complete market. Thirdly, the required options for portfolio insurance are not always available, as also Rubinstein (1985) argues. Thus, we extend previous studies by comparing the performance of the two strategies when implemented with delta replication\(^4\) in discrete time in two settings: when the market follows a geometric Brownian diffusion and when the market is discontinuous, modelling jumps via a time-changed geometric Brownian process. We replicate the desired payoffs at regular time intervals and evaluate the expected utilities of the final payoffs without focusing especially on any gap below the desired floor.

The paper has the following structure. In Section 2 we specify the continuous and discontinuous price dynamics for a risky asset price and apply them to portfolio insurance strategies. Section 3 introduces a calibration method for a simple time-changed Brownian and illustrates its application on two sets of data: a single share price and an equity index. Despite the simplicity of this process we show it fits empirical returns much better than a geometric Brownian process. Section 4 discusses optimal payoff profiles for different time horizons and investors with diverse sensitivities of risk tolerance to wealth. We conclude and comment on the merits of CPPI strategies in Section 5.

2 Model Specification

For modelling the risky asset price we first use the standard geometric Brownian \((2)\). Whilst the drift and volatility are assumed constant, the analysis could be easily extended to deterministic drift \(\mu(t)\) and volatility \(\sigma(t)\). In theory, in a complete market with continuous price processes,

\(^{4}\text{For the replication of a synthetic option and the constant proportion approach see also e.g. Zhu and Kavee (1988).}\)
standard option and power option payoffs could be perfectly replicated with continuous dynamic investment strategies in the underlying risky asset and in the risk-free asset. In particular, there would be no risk of falling below the floors of these profiles. But continuous replication would imply an infinite transaction volume and cost, and is therefore impossible. Market prices are subject to jumps. So, in practice discrete time investment strategies and discontinuous price processes lead to imperfect replications.

There is ample evidence for the existence of price jumps, if only because most markets trade during limited time periods every day. Discontinuous returns were modelled by Merton (1976) by adding a Poisson-driven jump term for rare large moves to a standard geometric Brownian process. This class of jump-diffusion processes is now very widely used to account for discontinuous returns in financial assets. As an alternative, Madan and Seneta (1987) introduced time-changed Lévy processes to model long tailed stock returns. The authors consider pure jump processes that allow small moves to occur with a higher probability than large moves. This is a generalisation of the results of Clark (1973) who introduced subordinated processes that make use of a random time-change for the evaluation of a geometric Brownian motion. The time jump process and the Brownian are taken to be independent. The main advantage of the time-change approach is that the price can still jump upwards or downwards, but the geometric process prohibits negative values for the risky asset after the occurrence of a jump.

Time change modelling is empirically supported by previous research. For instance Geman and Ané (1996) showed that calendar time returns are not normally distributed, but that returns on a unit trade basis follow a normal distribution. Madan et al. (1998) confirmed that a Brownian motion time-changed process that models time with gamma distributed jumps fits historical returns significantly better than standard diffusion models. Madan and Seneta (1990), Madan and Milne (1991) and others consider time-changed processes without a drift resulting in a process without skewness for log-returns. The general case with skewness is treated in Madan et al. (1998) and Geman and Madan (2001).

For our discontinuous price modelling we adopt a simple time-changed Brownian, which we label the ‘discontinuous return’ model, with the following asset price dynamics:

$$dS(t) = S(t)(\mu \, dt + \sigma \, dW(t(1 + \delta n))).$$

(4)

where $$\delta$$ is a constant time interval and $$n$$ is the number of jumps over time interval $$[0,t]$$, assumed to follow a Poisson distribution with intensity parameter $$\nu$$, the expected number of jumps per unit time.

The pricing of standard options under (2) is straightforward with the closed form solution of
Black and Scholes (1973). With discontinuous returns the financial market is incomplete and theoretically there is no unique risk-neutral probability measure to price options. We follow the approach of Merton (1976) and argue that jump risk is diversifiable. So, we price in a risk neutral world with risk-free rate $r^f$ where $\mathbb{E}[S(t) \exp(-r^f t)] = S(0)$, that is with a modified drift term to accommodate for the time jumps, and obtain:

$$S(t) = S(0) \exp \left( r^f t - \frac{1}{2} \sigma^2 t(1 + \delta n) + \sigma W(t(1 + \delta n)) \right).$$

But with (5) perfect option replication is no longer feasible even in continuous time. There is a residual replication error, similar to the replication error generated by discrete time replication strategies. Moreover the replication error and cost are path-dependent and there is no closed form solution for the probability distribution of returns. Using Merton’s argument, the fair price of options is a mixture of option prices conditional on the number of jumps.

For performance analysis, we use a maximum expected utility criterion. Investors have individual utility functions to represent their risk preferences and we determine certainty equivalent returns (CER) via simulation to assess the different investment strategies. We adopt the general class of hyperbolic absolute risk aversion (HARA) utility functions:

$$u(w) = -\left(1 + \frac{\eta}{\lambda}(w - w(0))\right)^{1-\frac{1}{\eta}} \frac{1}{1 - \eta}, \text{ with } (w - w(0)) \geq \frac{-\lambda}{\eta} \text{ and } \eta \geq 0,$$

where $w$ represents the present value of future wealth; the local coefficient of absolute risk tolerance, $\lambda$, is defined as a proportion of initial wealth $w(0)^5$ and $\eta$ is the sensitivity of risk tolerance to wealth. For $\eta \downarrow 0$ a HARA utility function converges to a negative exponential utility function which exhibits constant absolute risk aversion (CARA) $\frac{1}{\lambda}$ and for $\eta = 1$ it is a displaced logarithmic utility function, exhibiting constant relative risk aversion (CRRA).

Leland (1979), Brennan and Solanki (1981), and Constantinides (1982) derive a general condition for the maximum expected utility of present value of wealth function $w$ for an investor with utility function $u(w)$ given the investor’s forecast probability $p(r)$ for the excess return $r$ and risk neutral probability $q(r)$. They show that the first order derivative $u_w(w)$ must be proportional to $q(r)/p(r)$. Pézier (2011b) gives closed form solutions for optimal payoffs and optimal CERs using HARA utilities and normal and log-normal probability distributions. With a HARA utility function (6), a normally distributed excess log-return $r \sim N(\mu - r^f, \sigma^2)$ and the corresponding risk-neutral distribution $r \sim N(0, \sigma^2)$, the optimality condition yields:

$$[1 + \frac{\eta}{\lambda}(w - w(0))]^{-\frac{1}{\eta}} \propto \exp(-\frac{\mu - r^f}{\sigma^2} r).$$

\footnote{Without loss of generality we can choose $w(0) = 1.$}
Setting the initial condition \( E_Q[w] = 1 \), Pézier finds that the present value of the optimal payoff for an initial wealth of 1 and an investment horizon \( T \) is:

\[
w(T) = \left( 1 - \frac{\lambda}{\eta} \right) + \frac{\lambda}{\eta} \exp \left( m^* rT - \frac{1}{2} m^* \sigma^2 T \right)
\]

with \( m^* = \eta(\mu - r_f)/\sigma^2 \). This is a power payoff as a function of the underlying risky asset price (2) as produced by a CPPI strategy in continuous time. The optimal multiplier is \( m^* \) and the initial value of the optimal floor is:

\[
F^* = \left( 1 - \frac{\lambda}{\eta} \right).
\]

The curvature of the optimal payoff is

\[
\frac{w_{rr}}{w_r} = \frac{m^* - 1}{S^c(T)}
\]

so the payoff is linear for \( m^* = 1 \), convex if \( m^* > 1 \), and concave if \( 0 < m^* < 1 \). For a HARA utility the annualised CER over period \([0,T]\) of the optimal payoff is, for \( \eta > 0 \):

\[
CER^*(T) = \frac{1}{T} \ln \left( 1 + \frac{\lambda}{\eta} \left( \exp \left( \frac{1}{2} m^* (\mu - r_f) T \right) - 1 \right) \right)
\]

With a discontinuous risky asset price process there are no closed form solutions for the optimal payoff. The Black-Scholes formula and the corresponding fair pricing of a power payoff need to be adjusted to account for discontinuities in returns as described by Merton (1976). The stock price at time \( t \) is:

\[
S(t) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right) X_n.
\]

with the accumulated jump term \( X_n = \exp(-\frac{1}{2} \sigma^2 (t \delta n) + \sigma W(t \delta n)) \). We know from Merton (1976) that the option price can be expressed as a Poisson mixture. Define the Black-Scholes option prices with the variance \( \sigma_n^2 \), the risk-free rate \( r_n \) conditional on the number of jumps \( n \) and \( \tau \) the time to maturity as \( f_n(S, \tau) = BS(S, \tau, K, \sigma_n^2, r_n) \), then the option price under the discontinuous return diffusion can be written as:

\[
C(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) f_n(S, \tau)
\]

with \( P_n(\tau) = \exp(-\nu \tau) \frac{(\nu \tau)^n}{n!} \). The jump component has a variance of \( (\sigma^2 n \delta / \tau) \) and the conditional variance is therefore \( \sigma_n^2 = \sigma^2 (1 + n \delta / \tau) \). The drift in \( X_n \) is adjusted accordingly.

In Appendix A we show that CPPI strategies can also be priced with a Poisson mixture when
the underlying price process is a time-changed Brownian motion.

3 DATA AND CALIBRATION

We choose the S&P 500 index\textsuperscript{6} to represent an equity index and the Apple Computer stock to represent a single share price with higher volatility and more discontinuities. We calibrate the price dynamics of the risky assets from the 1 January 1990 to the 25 August 2010. This period covers a variety of events in the financial market starting with the longest economic expansion in US history in the 90s, but also the outbreak of two golf wars (1990 and 2003), the crash of LTCM in 1998, the dot.com bubble in the late 90s, the following stock market downturn and, most recently, the financial crisis of 2008-09.

The calibration of the standard Brownian motion model is straightforward. We simply estimate the drift and the volatility parameters of the log-return. Two more parameters, the size and the intensity of the time-jumps need be assessed for the time-changed process. The drift is fitted as in the continuous return model. We calibrate the other three parameters by fitting the first three even central moments $\mu_2$, $\mu_4$ and $\mu_6$ of the log-return.\textsuperscript{7} (for the derivation see Appendix B):

$$
\begin{align*}
\mu_2 &= \sigma^2(1 + \nu \delta) \tag{13} \\
\mu_4 &= 3\sigma^4(1 + 2\nu \delta + \nu(\nu + 1)\delta^2) \tag{14} \\
\mu_6 &= 15\sigma^6(1 + 3\nu \delta + 3\nu(\nu + 1)\delta^2 + (\nu^3 + 3\nu^2 + \nu)\delta^3) \tag{15}
\end{align*}
$$

On a time series of daily log-returns we equate the theoretical moments to the sample moments $\hat{\mu}_2$, $\hat{\mu}_4$ and $\hat{\mu}_6$. Setting $\chi = \nu \delta$ we obtain the following conditions:

$$
\begin{align*}
\hat{\mu}_2 &= \sigma^2(1 + \chi), \tag{16} \\
\frac{\hat{\mu}_4}{3\hat{\mu}_2^2} &= \frac{\chi \delta}{(1 + \chi)^2} + 1 \tag{17} \\
\frac{\hat{\mu}_6}{15\hat{\mu}_2^3} &= \frac{\chi \delta (3 + 3\chi + \delta)}{(1 + \chi)^3} + 1 \tag{18}
\end{align*}
$$

In Appendix B these are solved to find $\sigma$, $\delta$ and $\nu = \chi/\delta$ in terms of the sample moments. We fit the time-changed model to the index and the single stock daily log-returns and report the calibrated parameter values in Table 1. The risk-free rate $r^f$ is chosen as the average of the yield on 13-weeks US Treasury Bills over the respective time period.

\textsuperscript{6}In the following we refer to the S&P 500 just as the S&P.
\textsuperscript{7}The uneven central moments of the log-return are nil.
\textsuperscript{8}Abbreviations C and D denote the continuous and discontinuous models respectively.
Table 1: Annualised calibrated parameters for S&P and Apple daily log-returns

<table>
<thead>
<tr>
<th></th>
<th>$\gamma^{f}$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\delta$</th>
<th>$\nu$</th>
<th>$\chi/(1 + \chi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P C</td>
<td>3.625%</td>
<td>5.225%</td>
<td>18.663%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P D</td>
<td>3.625%</td>
<td>5.225%</td>
<td>14.662%</td>
<td>0.050</td>
<td>12.500</td>
<td>0.383</td>
</tr>
<tr>
<td>Apple C</td>
<td>3.625%</td>
<td>9.047%</td>
<td>54.998%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Apple D</td>
<td>3.625%</td>
<td>9.047%</td>
<td>42.902%</td>
<td>0.486</td>
<td>1.325</td>
<td>0.391</td>
</tr>
</tbody>
</table>

Table 2: Historical and simulated returns for S&P and Apple daily log-returns

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\xi}$</th>
<th>$\hat{\kappa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historical</td>
<td>0.021%</td>
<td>1.176%</td>
<td>-0.200</td>
<td>8.857</td>
</tr>
<tr>
<td>S&amp;P C</td>
<td>0.020%</td>
<td>1.172%</td>
<td>0.004</td>
<td>-0.011</td>
</tr>
<tr>
<td>S&amp;P D</td>
<td>0.020%</td>
<td>1.182%</td>
<td>-0.012</td>
<td>8.738</td>
</tr>
<tr>
<td>Historical</td>
<td>0.036%</td>
<td>3.465%</td>
<td>-4.195</td>
<td>87.437</td>
</tr>
<tr>
<td>Apple C</td>
<td>0.036%</td>
<td>3.471%</td>
<td>-0.001</td>
<td>-0.021</td>
</tr>
<tr>
<td>Apple D</td>
<td>0.038%</td>
<td>3.530%</td>
<td>-1.628</td>
<td>91.269</td>
</tr>
</tbody>
</table>

Table 1 shows that in the given period the S&P 500 exhibits a lower return and a lower standard deviation than the Apple stock. Time jumps in the S&P 500 data occur at a higher rate than in the Apple data, but their size is smaller than for Apple. The volatility parameters are different in the continuous and discontinuous models as $\sigma$ is constant for continuous returns but depends on the size and intensity of jumps for discontinuous returns. For the S&P we find that the volatility on a jump day is equivalent to the volatility of 13.6 non-jump days, but for Apple it is equivalent to the volatility of 123.4 non-jump days. The factor $\chi/(1 + \chi)$ in the last column indicates the fraction of total variance generated by the jump process, in both cases this fraction is close to 40%.

In Table 2 we assess simulation errors by comparing the mean $\mu$, the standard deviation $\sigma$, the skewness $\xi$ and the excess kurtosis $\kappa$ of historical and simulated returns using 100,000 sample paths. The simulations with jumps match not only the first and second moments nearly as well as the diffusion model but give a much better match of the historical daily excess kurtosis. The skewness terms in the simulations are entirely attributable to sampling error since all odd central moments should be zero according to both models, whereas the historical skewness is negative for Apple.
4 Optimal Payoffs

We optimise the strategy performance and find parameter sets \( (m^*, F^*) \) and \( (K^*, F^*) \) for CPPI and OBPI respectively, by maximising the corresponding certainty equivalent returns. Like Benninga and Blume (1985), Bird et al. (1990) and Dichtl and Drobeta (2011) amongst other, we perform stochastic scenario simulations as the theoretical distributions are not tractable and solutions cannot be found analytically in most cases. The exception is the continuous return model under continuous replication; in this case we compare the theoretical results for CPPI given by (10) with results from simulations. We consider short (0.5 years), medium (5 years) and long-term (20 years) horizons and evaluate theoretical and replicated payoffs. We rebalance the portfolio 60 times over the investment horizon in all cases. We also vary the parameters of the HARA utility function to account for investors with diverse sensitivities of risk tolerance to wealth.

We simulate one set of 100,000 daily return paths for each of the three time horizons and for each of the S&P index and the Apple stock. To check how simulation errors influence the results, we bootstrap 1,000 such sets and construct confidence intervals for the certainty equivalent returns.

4.1 Strategy evaluation with fair-value payoffs: Continuous Prices

Table 3 shows optimal annualised CPPI CERs and the corresponding strategy parameters as obtained from the theoretical results in Section 2. We use equation (10) for the optimal certainty equivalent excess return \( \text{CER}^* \) in % per annum with \( m^* \) as the optimal multiplier and the floor \( F^* \) given by (8). Multiplier and floor are independent of the investment horizon. \( \text{CER}^* \)s increase with \( T \) when the floor and the multiplier are positive. Without loss of generality, the initial wealth \( w(0) \) is set to 1 and the coefficient of local risk aversion \( \lambda \) in the HARA utility function is set at a typical value of 20% of initial wealth (Basky et al. (1997)).

With equation (9) and the calibrated parameters from Table 1 the investor’s optimal payoff function on the S&P is convex for \( \eta > 2.177 \) and concave for lower levels of \( \eta \).\(^9\) We consider \( \eta = 3 \) and 1 in order to illustrate both convex and concave optimal payoffs. For Apple the payoff becomes convex for \( \eta > 5.579 \) as the \( \frac{\mu - r_f}{\lambda} \) ratio is smaller and we therefore consider \( \eta = \frac{\mu - r_f}{\lambda} \)

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\(^9\) Hwang and Satchell (2010) show that loss aversion is higher than usually assumed in the literature and therefore our choice of \( \lambda \) is very conservative. Different levels of \( \lambda \) would only influence the floor but neither the multiplier nor the strike. By choosing \( \lambda \) relative to the initial wealth also the level of \( w(0) \) does not influence the results. Otherwise with constant absolute \( \lambda \) changing the initial wealth to \( (c \ w(0)) \) would give \( (c \ \text{CER}^*) \) and the parameter sets would shift to \( (m^*, c \ F^*) \) and \( (K^*, c \ F^*) \) for CPPI and OBPI respectively.

\(^10\) Assets with larger values of \( \frac{\mu - r_f}{\sigma^2} \) would lead to convex payoffs for lower \( \eta \) and make the purchase of portfolio insurance more desirable.
Table 3: Theoretical optimal strategies: Theoretical Payoff, Continuous Prices

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P</th>
<th></th>
<th>Apple</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPPI&lt;sub&gt;η=3&lt;/sub&gt;</td>
<td>CPPI&lt;sub&gt;η=1&lt;/sub&gt;</td>
<td>CPPI&lt;sub&gt;η=6&lt;/sub&gt;</td>
<td>CPPI&lt;sub&gt;η=3&lt;/sub&gt;</td>
</tr>
<tr>
<td>0.5 years</td>
<td>0.074</td>
<td>0.074</td>
<td>0.098</td>
<td>0.098</td>
</tr>
<tr>
<td>5 years</td>
<td>CER*</td>
<td>0.075</td>
<td>0.074</td>
<td>0.104</td>
</tr>
<tr>
<td>20 years</td>
<td>m*</td>
<td>0.082</td>
<td>0.076</td>
<td>0.130</td>
</tr>
<tr>
<td></td>
<td>F*</td>
<td>1.378</td>
<td>0.459</td>
<td>1.076</td>
</tr>
</tbody>
</table>

Table 4: Simulated optimal strategies: Theoretical Payoff, Continuous Prices, S&P

<table>
<thead>
<tr>
<th></th>
<th>CPPI&lt;sub&gt;η=3&lt;/sub&gt;</th>
<th>OBPI&lt;sub&gt;η=3&lt;/sub&gt;</th>
<th>CPPI&lt;sub&gt;η=1&lt;/sub&gt;</th>
<th>OBPI&lt;sub&gt;η=1&lt;/sub&gt;</th>
<th>CPPI&lt;sub&gt;η=0&lt;/sub&gt;</th>
<th>OBPI&lt;sub&gt;η=0&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>CER*</td>
<td>0.067</td>
<td>0.066</td>
<td>0.066</td>
<td>0.066</td>
<td>0.066</td>
</tr>
<tr>
<td></td>
<td>m*/K*</td>
<td>1.817</td>
<td>0.715</td>
<td>0.798</td>
<td>0.000</td>
<td>0.300</td>
</tr>
<tr>
<td></td>
<td>F*</td>
<td>0.952</td>
<td>0.974</td>
<td>0.891</td>
<td>0.913</td>
<td>0.710</td>
</tr>
<tr>
<td>5 years</td>
<td>CER*</td>
<td>0.074</td>
<td>0.073</td>
<td>0.072</td>
<td>0.070</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>m*/K*</td>
<td>1.432</td>
<td>0.507</td>
<td>0.477</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>F*</td>
<td>0.936</td>
<td>0.949</td>
<td>0.811</td>
<td>0.910</td>
<td>−∞</td>
</tr>
<tr>
<td>20 years</td>
<td>CER*</td>
<td>0.082</td>
<td>0.080</td>
<td>0.075</td>
<td>0.069</td>
<td>0.072</td>
</tr>
<tr>
<td></td>
<td>m*/K*</td>
<td>1.414</td>
<td>0.757</td>
<td>0.460</td>
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<tr>
<td></td>
<td>F*</td>
<td>0.934</td>
<td>0.941</td>
<td>0.802</td>
<td>0.910</td>
<td>−∞</td>
</tr>
</tbody>
</table>

6 and 3. Investors with high levels of η, that is rapidly decreasing levels of risk tolerance with decreasing wealth (vice versa, rapidly increasing risk tolerance with increasing wealth) prefer high floor levels in combination with convex payoffs, giving them a great upside potential. For lower η’s floor and multiplier decrease. Investors with lower η see relatively higher utility in lower payoffs and less utility in higher payoffs; they prefer concave payoffs.

Tables 4 and 5 report the simulation results for continuous returns, indicating maximal annualised CER* and the corresponding optimal parameter sets. Table 4 is for the S&P index and Table 5 is for the Apple stock.

In Table 4 we report the results obtained from a single set of 100,000 simulations. Investors who prefer convex payoff functions have a high probability of getting close to the floor but also of gaining very high returns. We evaluate the theoretical portfolio value at maturity only since the theoretical payoff is path-independent. Strategies with high multipliers have a positively skewed
wealth distribution with positive excess kurtosis. With simulations these results are obtained from a reduced sample as many simulation paths hit the floor. For investors with $\eta = 3$ and a 20-year investment horizon, we check the accuracy of the results with bootstrapping and estimate the 5% confidence intervals. These confidence intervals for CPPI\textsubscript{$\eta=3$} and OBPI\textsubscript{$\eta=3$} are: $P(0.0821 < CE_{\text{CPPI}} < 0.0823) = 0.05$ and $P(0.0801 < CE_{\text{OBPI}} < 0.0803) = 0.05$. The means of the bootstrapped strategy parameters ($m^*, F^*$) are (1.380, 0.933) and (0.767, 0.941), respectively.

As the investment horizon increases, the results from simulations in Table 4 converge towards the theoretical values in Table 3. For short-term horizons there are larger inaccuracies due to the limited spectrum of sample path outcomes, but for long-term horizons the simulated results are accurate. We verify with bootstrapping that the simulated strategy parameters are on average equal to the theoretical ones. The precision level in the floor is higher than in the multiplier. In summary, the simulated results are most significant for long-term horizons and we shall focus on the 20-year horizon to draw our main conclusions.

Table 4 confirms the outperformance of CPPI power payoffs over OBPI payoffs: the CER\textsuperscript{*}s from CPPI strategies are, in all cases, greater than or equal to those from OBPI strategies. The multipliers for CPPI\textsubscript{$\eta=3$} are greater than 1; the investor is a portfolio insurance buyer. For CPPI\textsubscript{$\eta=1$} and CPPI\textsubscript{$\eta=0$} we obtain the anticipated concave payoffs with multipliers below 1: the investor would therefore sell portfolio insurance. The lower $\eta$, the higher the concavity of the payoff profile is. The annualised CER\textsuperscript{*}s increase with the investment horizons as expected.

$F^*$ also evolves as expected from the theory; the lower $\eta$, the lower the floor is. Investors with an exponential utility ($\eta = 0$) and a 5-year or 20-year term horizon set their $F^*$ to minus infinity (no floor) in combination with $m^* = 0$. We report these results using smaller characters. Theoretically we should observe no floor even for the shorter time horizons, but in the short-term, extremely low price variations are too rare to cover the whole spectrum of possible outcomes and the optimisation does not quite converge to the theoretical result.

OBPI strategies always produce convex payoffs; they become linear only when the call strike is nil. The higher the strike of the call option, the cheaper it becomes and the more options the investor can buy. The purchase of high strike calls therefore corresponds to more convex OBPI payoffs. $K^*$ for OBPI\textsubscript{$\eta=3$} is positive and becomes zero for OBPI\textsubscript{$\eta=1$} and OBPI\textsubscript{$\eta=0$} as the investor would prefer a concave payoff which cannot be provided by a long call option. Consequently, the reported OBPI CER is constrained; we report these results in smaller characters. $F^*$ increases with rising $\eta$.

Table 5 presents the corresponding results for the Apple stock and in most cases CPPI signifi-
cantly outperform OBPI. The Apple CPPI CER’s exceed the corresponding S&P CER’s from Table 4 in all cases. The same holds for OBPI except when \( \eta \) is low. This shows that CPPI can better accommodate riskier assets than OBPI and that even very risk averse investors can gain from investing in risky assets.

The higher volatility of the underlying price increases the outperformance of CPPI over OBPI but also increases the variability of the results. This is especially evident for investors with high \( \eta \) as they prefer high multipliers which give them great upside potential. The variability of the OBPI results is much lower as standard options offer a more limited leverage than power payoffs. We check the accuracy of our results with bootstrapping for a 20-year horizon and an investor with \( \eta = 6 \) and find the results to be significant but with much wider confidence intervals than for the S&P.\(^\text{11}\)

The difference between the theoretical results in Table 3 and the simulated results in Table 5 are greater for Apple than they are between Table 3 and 4 for the S&P, because of the greater volatility of Apple. Nonetheless, the simulation results converge to their theoretical values over long time horizons. Focusing on the 20-year horizon we verify that the multipliers for Apple are smaller than for S&P because of the smaller ratio \( \left( \frac{\mu - r}{\sigma^2} \right) \) Apple’s return. Optimal floor levels are independent of the choice of risky asset.

OBPI strategies show the investor’s desire for a concave payoff for \( \eta = 3 \) and \( \eta = 0 \); the optimal

\(^{11}\) We find that the 5% confidence intervals for CPPI\( \eta = 6 \) and OBPI\( \eta = 6 \) are wider for Apple than S&P with \( \eta = 3 \) as Apple has a higher volatility and the sensitivity of risk tolerance to wealth is larger: \( P(0.1322 < C_{E_{CPPI}} < 0.2916) = 0.05 \) and \( P(0.1305 < C_{E_{OBPI}} < 0.1315) = 0.05 \). Again we find that the confidence intervals do not overlap and conclude that the results are significant. The confidence interval for CPPI is much wider than the one for OBPI as extreme returns can be gained under CPPI with a highly volatile underlying.
strike is nil or near zero, only investors with $\eta = 6$ choose positive strikes. Floor levels also decrease with lower $\eta$. The restriction to non-negative strikes decreases CER’s for OBPI$_{\eta=6}$ and OBPI$_{\eta=3}$ for longer time horizons. OBPI investors are limited in their parameter choice and do not gain from longer investment horizons.

4.2 Strategy evaluation with delta replication: Continuous Prices

In the previous sub-section we optimised the expected utility of a theoretical payoff at maturity. But if investors have to replicate the payoff, the value of the strategy depends on the final, path-dependent, wealth generated by the replication and not on the theoretical path-independent payoff at maturity. The path-dependency introduced by discrete replication heavily penalises strategies that hit the floor before maturity, which is more likely with strategies offering high leverage on volatile assets. Investors may therefore prefer strategies offering less leverage. Correspondingly, we expect minor decreases in the CER’s and the effect to be more significant for Apple than for S&P. As we choose to rebalance the replication portfolio 60 times for each investment horizon, the effects of discrete replication should be less in the shorter-term settings than for the 20-year time horizon.

We replicate the investment strategies using discrete delta hedging. For OBPI we use the call delta from the Black-Scholes formula. The sensitivity of the CPPI payoff to changes in the underlying is:

$$\Delta_{CPPI}(S^c, t) = \frac{m}{S^c} C_{CPPI}(S^c, t). \quad (19)$$

With a HARA utility function, the optimal floor of a CPPI strategy is located at the point were the utility becomes infinitely negative. In continuous time, the probability of reaching the floor is nil, but with discrete replication there is a finite probability of falling below the floor. To improve the stability of the results when seeking the optimal payoff, we set a minimum negative utility. This low cut-off has no material effect on the parameters of the optimal payoff. In practice, it can also be argued that no human institution can provide absolute guarantees. This is recognised, for example, in the Basel minimum capital requirement regulations for banks. The minimum level of capital requirement is set to ensure a probability of default of no more than about 0.1% per year, that is approximately the empirical default frequency of ’A’ rated firms.

We report the results for S&P in Table 6 which should be compared to Table 4; they confirm

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12For option replication see Rubinstein and Leland (1981) and Harrison and Pliska (1981). A small improvement could be obtained by choosing a minimum variance delta over the delta revision interval rather than the instantaneous delta.

13See Bertrand and Prigent (2005).
Table 6: Simulated optimal strategies: Replicated Payoff, Continuous Prices, S&P

<table>
<thead>
<tr>
<th></th>
<th>CPPI$_{t=3}$</th>
<th>OBPI$_{t=3}$</th>
<th>CPPI$_{t=1}$</th>
<th>OBPI$_{t=1}$</th>
<th>CPPI$_{t=0}$</th>
<th>OBPI$_{t=0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER$^*$</td>
<td>0.067</td>
<td>0.066</td>
<td>0.066</td>
<td>0.066</td>
<td>0.066</td>
<td>0.066</td>
</tr>
<tr>
<td>$m^<em>/K^</em>$</td>
<td>1.869</td>
<td>0.722</td>
<td>0.850</td>
<td>0.000</td>
<td>0.353</td>
<td>0.000</td>
</tr>
<tr>
<td>$F^*$</td>
<td>0.953</td>
<td>0.975</td>
<td>0.900</td>
<td>0.913</td>
<td>0.754</td>
<td>0.914</td>
</tr>
<tr>
<td>5 years</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>CER$^*$</td>
<td>0.074</td>
<td>0.073</td>
<td>0.072</td>
<td>0.070</td>
<td>0.071</td>
<td>0.065</td>
</tr>
<tr>
<td>$m^<em>/K^</em>$</td>
<td>1.435</td>
<td>0.519</td>
<td>0.482</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$F^*$</td>
<td>0.937</td>
<td>0.950</td>
<td>0.813</td>
<td>0.910</td>
<td>$-\infty$</td>
<td>0.917</td>
</tr>
<tr>
<td>20 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER$^*$</td>
<td>0.082</td>
<td>0.079</td>
<td>0.075</td>
<td>0.069</td>
<td>0.072</td>
<td>0.054</td>
</tr>
<tr>
<td>$m^<em>/K^</em>$</td>
<td>1.355</td>
<td>0.839</td>
<td>0.461</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$F^*$</td>
<td>0.935</td>
<td>0.942</td>
<td>0.803</td>
<td>0.910</td>
<td>$-\infty$</td>
<td>0.931</td>
</tr>
</tbody>
</table>

our expectations. With continuous returns the discrete replication is very accurate and we find only a minor deterioration of performance. CPPI strategies still outperform OBPI strategies. In this setting the replication error is not large enough to cause a visible deterioration in the CER$^*$ at the half year and 5-year time horizons. Of course, we would obtain a greater reduction in CER$^*$ if we were to take into account transaction costs, but likewise, in the alternative we would have to take into account the profit margin of the option market maker underwriting a theoretical payoff.

The floors are hardly affected but are generally slightly raised to ensure greater protection. Multipliers and strikes are generally increased for the short and medium-term, but decreased slightly for long-term investments and investors with convex payoff profiles. The changes in option strikes have the same effects on the payoff curvature as the changes in the multipliers of CPPI strategies. As the previous simulation results for the theoretical payoffs were shown to be significant, we conclude that they must also be significant in this replication setting.

We report the results for Apple in Table 7, which should be compared to Table 5. The performance deterioration caused by discrete replication is more noticeable than for the lower volatility S&P. It is also more noticeable for longer-term investments (larger rebalancing intervals) and payoffs with greater curvature, as one would expect.

Again, floors mostly increase to protect a greater fraction of the portfolio wealth. Multipliers and strikes now decrease significantly for investors with convex payoff profiles and medium and long-term investment horizons. OBPI strategies offer less leverage than CPPI strategies and therefore a lesser probability of hitting the floor during discrete replication. Option based strategies consequently suffer less from discrete replication. But as a result from parameter
Table 7: Simulated optimal strategies: Replicated Payoff, Continuous Prices, Apple

<table>
<thead>
<tr>
<th></th>
<th>CPPL(\eta=6)</th>
<th>OBPI(\eta=6)</th>
<th>CPPL(\eta=3)</th>
<th>OBPI(\eta=3)</th>
<th>CPPL(\eta=0)</th>
<th>OBPI(\eta=0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>CER(^*)</td>
<td>0.091</td>
<td>0.091</td>
<td>0.089</td>
<td>0.089</td>
<td>0.088</td>
</tr>
<tr>
<td></td>
<td>(m^<em>/K^</em>)</td>
<td>1.310</td>
<td>0.345</td>
<td>0.682</td>
<td>0.050</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>(F^*)</td>
<td>0.973</td>
<td>0.977</td>
<td>0.950</td>
<td>0.967</td>
<td>0.658</td>
</tr>
<tr>
<td>5 years</td>
<td>CER(^*)</td>
<td>0.107</td>
<td>0.105</td>
<td>0.098</td>
<td>0.088</td>
<td>0.093</td>
</tr>
<tr>
<td></td>
<td>(m^<em>/K^</em>)</td>
<td>1.111</td>
<td>0.000</td>
<td>0.558</td>
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</tr>
<tr>
<td></td>
<td>(F^*)</td>
<td>0.968</td>
<td>0.967</td>
<td>0.936</td>
<td>0.964</td>
<td>(-\infty)</td>
</tr>
<tr>
<td>20 years</td>
<td>CER(^*)</td>
<td>0.147</td>
<td>0.144</td>
<td>0.108</td>
<td>0.080</td>
<td>0.091</td>
</tr>
<tr>
<td></td>
<td>(m^<em>/K^</em>)</td>
<td>1.001</td>
<td>0.000</td>
<td>0.590</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>(F^*)</td>
<td>0.967</td>
<td>0.967</td>
<td>0.943</td>
<td>0.960</td>
<td>(-\infty)</td>
</tr>
</tbody>
</table>

adjustments, more OBPI strategies are affected by the restriction to positive option strikes.

For both underlyings and both strategies, the multipliers and strikes affecting the curvature are adjusted to generate more linear payoffs with discrete replication. The adjustments of the multiplier and the strike influence all moments of the portfolio value distribution; higher parameters lead to larger higher moments. Increasing (decreasing) the floor balances the higher (lower) volatility but has no effect on the higher moments.

4.3 Strategy evaluation with fair-value payoffs: Discontinuous Prices

We now consider discontinuous returns with the time-changed geometric Brownian as defined in (4). As before we seek the maximal certainty equivalent returns and optimal sets \((m^*, F^*)\) and \((K^*, F^*)\). Discontinuous prices should have qualitatively similar effects to the jumps observed with discrete replication in the previous sub-section. Quantitatively, the effects should be relatively small when there are potentially many small jumps during the investment period, that is the case for S&P over long investment periods, but relatively larger when there are potentially few large jumps over the investment period, that is the case for Apple over short investment periods. We expect the strategy parameters to behave similarly as in the sub-section 4.2.

Tables 8 and 9 report the performances of CPPI and OBPI for the S&P and Apple, respectively. CPPI strategies still outperform OBPI in most scenarios but the performance is reduced by the introduction of jumps as we see from the comparison with Table 4. The performance degradation is, as expected, greater for short than for long-term investments and greater with
Table 8: Simulated optimal strategies: Theoretical Payoff, Discontinuous Prices, S&P

<table>
<thead>
<tr>
<th></th>
<th>CPPI_{η=3}</th>
<th>OBPI_{η=3}</th>
<th>CPPI_{η=1}</th>
<th>OBPI_{η=1}</th>
<th>CPPI_{η=0}</th>
<th>OBPI_{η=0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.064</td>
<td>0.064</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
</tr>
<tr>
<td>m*/K*</td>
<td>2.763</td>
<td>0.810</td>
<td>1.497</td>
<td>0.792</td>
<td>1.018</td>
<td>0.790</td>
</tr>
<tr>
<td>F*</td>
<td>0.969</td>
<td>0.982</td>
<td>0.944</td>
<td>0.981</td>
<td>0.918</td>
<td>0.981</td>
</tr>
<tr>
<td>5 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.073</td>
<td>0.073</td>
<td>0.070</td>
<td>0.069</td>
<td>0.069</td>
<td>0.065</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.475</td>
<td>0.625</td>
<td>0.579</td>
<td>0.438</td>
<td>0.177</td>
<td>0.387</td>
</tr>
<tr>
<td>F*</td>
<td>0.938</td>
<td>0.955</td>
<td>0.846</td>
<td>0.943</td>
<td>0.500</td>
<td>0.944</td>
</tr>
<tr>
<td>20 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.082</td>
<td>0.079</td>
<td>0.074</td>
<td>0.069</td>
<td>0.070</td>
<td>0.054</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.410</td>
<td>0.839</td>
<td>0.519</td>
<td>0.000</td>
<td>0.118</td>
<td>0.000</td>
</tr>
<tr>
<td>F*</td>
<td>0.934</td>
<td>0.942</td>
<td>0.825</td>
<td>0.910</td>
<td>0.251</td>
<td>0.930</td>
</tr>
</tbody>
</table>

Apple than with S&P.

Comparing the parameters in the continuous case to the discontinuous case we also find our expectation on the parameter behaviour confirmed. For a short-term investment the strategy parameters typically increase and they decrease for 5 and 20-year investments. This shows that there is a theoretical optimal level for the higher moments of the strategy wealth distribution. They much increase through the introduction of discontinuous returns. The decrease of the multipliers and the strikes counters an excessive increase in skewness, kurtosis and higher moments.

We also find that the evaluation of the theoretical payoff leads investors with high η’s to increase their optimal multipliers to unrealistic levels for a short-term investment horizon. Utility functions which are nearly linear on the high end make the investor prefer very high returns. As losses are at the same time limited to the floor investors see nearly unlimited return opportunities by increasing the payoff convexity. Consequently we do not report the results for those CPPI investors and again stress the necessity to replicate strategies and make them path-dependent to get realistic evaluations. We do not observe this problem for OBPI as the option strategy has a more limited leverage than power payoffs.

4.4 Strategy evaluation with delta replication: Discontinuous Prices

We replicate the payoffs as in Sub-section 4.2 but adjust the replication to the derivative pricing under discontinuous returns. Appendix C reports the technical details. We expect to see small effects for the low-volatility, small jumps asset in the short-term and large performance
deterioration for the high-volatility high jumps asset up to the 20-year horizon as in Sub-section 4.2.

The CPPI strategies still outperform matching OBPI strategies (compare Table 10 to Table 8). The losses from discrete rebalancing are larger than in Sub-section 4.2 as perfect replication becomes more difficult with discontinuous returns. As expected the effects are least evident in the short-term investment with its higher rebalancing frequency. Floors, multipliers and strikes tend to decrease for all investment horizons. This pattern, which we first saw in Sub-section 4.3, is amplified by the additional uncertainty generated by discrete rebalancing and applies now to all settings. That also results in more OBPI strategies moving to the linear payoff limit.

For the Apple stock results are displayed in Table 11 which, compared to Table 9, tells a similar story: CPPI strategies are at least as good as OBPI strategies in all cases. With the high volatility in the underlying and the larger jumps, manufacturing the final payoff is more difficult. The performance is inferior with discrete replication, especially at the 20-year horizon.

For shorter-term investments, the discrete replication strategy leads to less extreme returns, even for investors with high $\eta$. The use of very high multipliers, as an investor with $\eta = 6$ would have preferred in a short-term horizon in Sub-section 4.3, is not feasible as investors would risk falling to the floor very quickly. Thus, with discrete replication we can find stable values for CPPI$_{\eta=6}$ at all time horizons whereas our results in Section 4.3 were unstable for CPPI$_{\eta=6}$ for Apple at the half year horizon.

Comparing with Table 9 we find again that floors generally decrease. The same holds for multipliers and option strikes, but now most OBPI strategies cannot reach optimal levels as
Table 10: Simulated optimal strategies: Replicated Payoff, Discontinuous Prices, S&P

<table>
<thead>
<tr>
<th></th>
<th>CPPI_{η=3}</th>
<th>OBPI_{η=3}</th>
<th>CPPI_{η=1}</th>
<th>OBPI_{η=1}</th>
<th>CPPI_{η=0}</th>
<th>OBPI_{η=0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.064</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
<td>0.062</td>
</tr>
<tr>
<td>m*/K*</td>
<td>2.679</td>
<td>0.322</td>
<td>1.725</td>
<td>0.484</td>
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<td>0.326</td>
</tr>
<tr>
<td>F*</td>
<td>0.968</td>
<td>0.943</td>
<td>0.951</td>
<td>0.956</td>
<td>0.933</td>
<td>0.943</td>
</tr>
<tr>
<td>5 years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.073</td>
<td>0.072</td>
<td>0.070</td>
<td>0.069</td>
<td>0.069</td>
<td>0.065</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.481</td>
<td>0.531</td>
<td>0.554</td>
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<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>F*</td>
<td>0.939</td>
<td>0.951</td>
<td>0.838</td>
<td>0.911</td>
<td>−∞</td>
<td>0.918</td>
</tr>
<tr>
<td>20 years</td>
<td></td>
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<td></td>
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<td></td>
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</tr>
<tr>
<td>CER*</td>
<td>0.078</td>
<td>0.077</td>
<td>0.074</td>
<td>0.069</td>
<td>0.071</td>
<td>0.054</td>
</tr>
<tr>
<td>m*/K*</td>
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<td>0.708</td>
<td>0.474</td>
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<td>0.000</td>
</tr>
<tr>
<td>F*</td>
<td>0.938</td>
<td>0.947</td>
<td>0.807</td>
<td>0.910</td>
<td>−∞</td>
<td>0.930</td>
</tr>
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</table>

Table 11: Simulated optimal strategies: Replicated Payoff, Discontinuous Prices, Apple

<table>
<thead>
<tr>
<th></th>
<th>CPPI_{η=6}</th>
<th>OBPI_{η=6}</th>
<th>CPPI_{η=3}</th>
<th>OBPI_{η=3}</th>
<th>CPPI_{η=0}</th>
<th>OBPI_{η=0}</th>
</tr>
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<tr>
<td>0.5 years</td>
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</tr>
<tr>
<td>CER*</td>
<td>0.087</td>
<td>0.087</td>
<td>0.082</td>
<td>0.081</td>
<td>0.077</td>
<td>0.074</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.395</td>
<td>0.430</td>
<td>1.362</td>
<td>0.413</td>
<td>0.492</td>
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<tr>
<td>F*</td>
<td>0.976</td>
<td>0.981</td>
<td>0.976</td>
<td>0.980</td>
<td>0.937</td>
<td>0.982</td>
</tr>
<tr>
<td>5 years</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>CER*</td>
<td>0.091</td>
<td>0.091</td>
<td>0.086</td>
<td>0.074</td>
<td>0.082</td>
<td>0.041</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.061</td>
<td>0.000</td>
<td>0.483</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>F*</td>
<td>0.968</td>
<td>0.967</td>
<td>0.931</td>
<td>0.966</td>
<td>−∞</td>
<td>0.983</td>
</tr>
<tr>
<td>20 years</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>CER*</td>
<td>0.118</td>
<td>0.118</td>
<td>0.093</td>
<td>0.060</td>
<td>0.081</td>
<td>0.012</td>
</tr>
<tr>
<td>m*/K*</td>
<td>1.000</td>
<td>0.000</td>
<td>0.492</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>F*</td>
<td>0.967</td>
<td>0.967</td>
<td>0.942</td>
<td>0.965</td>
<td>−∞</td>
<td>0.993</td>
</tr>
</tbody>
</table>
they are restricted to positive strikes.

In Appendix D we compare CPPI CER’s for optimal levels of the floor for continuous and discontinuous returns, and theoretical and replicated payoffs. The graphs summarise and support the results on the strategy behaviour of this section.

5 SUMMARY AND CONCLUSION

This paper compares the performance of two popular portfolio insurance strategies in a fair contest in which they are either both purchased at fair prices or both manufactured via discrete replication strategies. Compared with previous research, which focussed on stochastic dominance criteria, we assume risk averse investors endowed with two-parameter HARA utilities with a range of risk tolerance sensitivities to wealth. The comparison between OBPI and CPPI strategies is based on their maximum certainty equivalent returns. The payoff parameters are the optimal multipliers and floors for CPPI, and the strikes and floors for OBPI.

We examine continuous and discontinuous return processes and a variety of time horizons for the investment. Continuous returns are modelled with a geometric Brownian process whilst discontinuous returns are modelled with a time-changed Brownian process fitting the first three even moments of the return distribution for two underlying assets: an equity index, the S&P 500, and a single share, Apple Computers. The combination of continuous or discontinuous return processes and payoffs that can be purchased at fair price or must be manufactured using discrete delta hedging creates four settings for the comparison of optimal OBPI and CPPI strategies.

In the first setting - with geometric Brownian returns and payoffs purchased at their fair price - theoretical results show that power payoffs, as produced by CPPI strategies, are optimal. But they may be convex or concave depending on whether the multiplier \( m = \eta(\mu - r_f)/\sigma^2 \) is greater or smaller than 1. We confirm this result by simulations and assess the accuracy of the parameters obtained through simulations with their theoretical values. The longer the time horizon, the more precise are the simulation results.

In the three other settings, theoretical results are not available but simulations show that CPPI strategies are never inferior to OBPI strategies, although the differences between certainty equivalent returns may be small, especially for low volatility assets.

Discrete replication has minor effects. Certainty equivalent returns are very slightly reduced whilst floors tend to be slightly increased. Multipliers and leverages are increased for short-term
investments in low volatility assets but decreased for long-term investments in riskier assets.

Accounting for discontinuous returns reduces the certainty equivalent returns, particularly for the Apple stock that has fewer and larger jumps than the S&P index, especially when the payoffs are discretely replicated. Multipliers and option strikes (and therefore leverages) need, generally, to be reduced for long-term investments.

We have not optimised the discrete delta replication strategies, taking into account the corresponding transaction costs, but Pézier (2011a) and Pézier and Vicedom (2011) show that the costs of optimal replication strategies are similar for standard and power options.

Power payoffs, as generated by CPPI strategies, are already favoured over OBPI strategies for long-term investments because of their open-endedness. They are also superior from an expected utility point of view for geometric Brownian price processes and HARA utilities. We have shown that they maintain this superiority in more realistic circumstances where the payoffs have to be discretely replicated and the underlying process is discontinuous.
A CPPI pricing with a time-changed Brownian

Merton (1976) gives the Black-Scholes-Merton differential equation for discontinuous returns that holds for any derivative:

\[
\frac{1}{2} \sigma^2 S^2 C_{SS} + r S C_S - C_T - r C + \lambda \mathbb{E}[C(S, \tau) - C(S^c, \tau)] = 0
\]

(A.1)

where subscripts denote partial derivatives, \(S^c\) is the continuous process:

\[
S^c(t) = S(0) \exp((\mu - \frac{1}{2} \sigma^2) t + \sigma W(t))
\]

(A.2)

and the discontinuous dynamics are:

\[
S(t) = S^c(t) X_n
\]

(A.3)

with the accumulated jump term \(X_n = \exp(-\frac{1}{2} \sigma^2 (t \delta n) + \sigma W(t \delta n))\)

Similarly to the option pricing formula under equation A.1 (also shown in Merton (1976)) the CPPI buffer payoff with discontinuous returns is\(^\text{14}\):

\[
C(S, t) = \sum_{n=0}^{\infty} P_n(t) \mathbb{E}_n[B(S^c X_n)]
\]

(A.4)

where \(B(S)\) is the CPPI buffer payoff as derived by Perold and Sharpe (1988). Therefore:

\[
C_S(S, t) = \sum_{n=0}^{\infty} P_n(t) \mathbb{E}_n[B_S(S^c X_n)X_n]
\]

\[
C_{SS}(S, t) = \sum_{n=0}^{\infty} P_n(t) \mathbb{E}_n[B_{SS}(S^c X_n)X_n^2]
\]

\[
C_t(S, t) = -\lambda F + \lambda \sum_{m=0}^{\infty} P_m(t) \mathbb{E}_{m+1}[B(S^c X_n)] + \sum_{n=0}^{\infty} P_n(t) \mathbb{E}_n[B_t(S^c X_n)].
\]

Substituting in (A.1) gives the following differential equation for the buffer:

\[
\sum_{n=0}^{\infty} P_n(t) \mathbb{E}_n[0.5 \sigma^2 S^2 X_n B_{SS} + r S X_n B_S - B_t - r B] + \lambda F - \lambda \sum_{m=0}^{\infty} P_m(t) \mathbb{E}_{m+1}[B(S^c X_n)]
\]

\[
= -\lambda \mathbb{E}[C(S, t) - C(S^c, t)]
\]

\(^\text{14}\)For the CPPI evaluation we use \(t\) as the elapsed time since initiation rather than \(\tau\) as time to maturity used in the Merton option pricing formula.
B Calibration of a time-changed Brownian

The random term of the time-changed Brownian per unit calendar time is \( \sigma W(1 + n\delta) \) with \( n \sim \text{Poisson}(\nu) \), that is therefore \( p(n) = \exp(-\nu)\nu^n/n! \) with constants \( \sigma, \nu \) and \( \delta \). Calibration is performed as follows: First, estimate the even moments \( \hat{\mu}_2, \hat{\mu}_4 \) and \( \hat{\mu}_6 \) on a time series of daily log-returns. Second, equate these estimators to the theoretical moments \( \mu_2, \mu_4 \) and \( \mu_6 \) of the time-changed Brownian: For a given \( n \), \( \sigma W(1 + n\delta) \) has a variance of \( \sigma^2 (1 + n\delta) \). With \( n \) random, the distribution of \( \sigma W(1 + n\delta) \) is a normal mixture. Its odd moments are nil and its even moments are (\( \pi \) denoting the expectation of \( n \)):

\[
\begin{align*}
\mu_2 &= \sum p(n) \sigma_n^2 = \sigma^2 (1 + \bar{n}\delta) \\
\mu_4 &= \sum p(n) 3\sigma_n^4 = 3\sigma^4 (1 + 2\bar{n}\delta + \bar{n}^2\delta^2) \\
\mu_6 &= \sum p(n) 15\sigma_n^6 = 15\sigma^6 (1 + 3\bar{n}\delta + 3\bar{n}^2\delta^2 + \bar{n}^3\delta^3)
\end{align*}
\]

With the Poisson distribution we have:

\[
\begin{align*}
\bar{n} &= \nu \\
\frac{\bar{n}(\bar{n}-1)}{\bar{n}(\bar{n}-1)(\bar{n}-2)} &= \nu^2 \implies \bar{n}^2 = \nu(\nu + 1) \\
\frac{\bar{n}(\bar{n}-1)(\bar{n}-2)}{\bar{n}(\bar{n}-1)(\bar{n}-2)(\bar{n}-3)} &= \nu^3 \implies \bar{n}^3 = \nu^3 + 3\nu^2 + \nu
\end{align*}
\]

Which yields (13) - (15). Equating these theoretical moments to the relevant estimators leads to (16) - (18) and setting \( A = \frac{\hat{\mu}_4}{(3\hat{\mu}_2^2)} - 1 \) and \( B = \frac{\hat{\mu}_6}{(15\hat{\mu}_2^2)} - 1 \) we obtain the process parameters:

\[
\begin{align*}
\chi &= \frac{1}{\frac{B}{A^2} - \frac{3}{4} - 1} \\
\sigma^2 &= \frac{\hat{\mu}_2}{1 + \chi} \\
\delta &= A(1 + \chi)^2 \frac{1}{\chi
\end{align*}
\]

C Option replication under time-changed Brownian

From (12) we obtain the standard option delta for replication with a Poisson mixture:

\[
\Delta(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) \Delta_n(S, \tau) \quad \text{(C.1)}
\]
with $\Delta_n(S, \tau) = df_n(S, \tau)/dS$. For optimisation in Matlab we vectorise the delta replication to increase calculation speed of\textsuperscript{15}. The delta conditional on $n$ jumps is:

$$
\Delta_n(S, \tau) = N\left( \frac{\ln(S/K) + (r + \sigma_n^2/2)\tau}{\sigma_n\sqrt{\tau}} \right)
$$

(C.2)

where $N$ is a standard normal distribution. Let $\mathbf{n} = [n_i]_{i=0,...,m}$ be a vector representing the number of jumps occurring until maturity and $\mathbf{t} = [\tau_j]_{j=0,...,T}$ the vector of the time to maturity at each rebalancing; using the outer product '$\otimes$' of these vectors and denoting $\mathbf{J}_{m,T}$ a $(m, T)$ matrix of ones, we write the variance matrix:

$$
\Sigma = \sigma^2 (\mathbf{J}_{m,T} + \delta (\mathbf{n} \otimes \mathbf{t}'))
$$

(C.3)

with elements $\sigma_{ij}$, $i = 1, ..., m$, $j = 1, ..., T$; the corresponding volatility matrix is $\mathbf{V} = [\sigma_{ij}^{1/2}]$. Furthermore define $\mathbf{S} = [S_j]_{j=0,...,T}$ as a simulated time series of stock prices at every rebalancing time and the auxiliary matrices $\mathbf{t}_1 = [\tau_j^{1/2}]$, $\mathbf{t}_2 = [\tau_j^{-1/2}]$, and $\mathbf{V}_1 = [\sigma_{ij}^{-1/2}]$. Using the term by term Hadamard product of the single matrix elements denoted '$\circ$', we obtain the matrix form of what is usually known as $d_1$ in the Black-Scholes option pricing formula:

$$
\mathbf{D} = \mathbf{J}_{m,1} \otimes \left[ \ln\left( \frac{1}{K} \cdot \mathbf{S} \right) \circ \mathbf{t}_1 \right] \circ \mathbf{V}_1 + (r \cdot \mathbf{V} + 1/2 \mathbf{V}) \circ (\mathbf{J}_{m,1} \otimes \mathbf{t}_2')
$$

(C.4)

From the elements $d_{ij}$ of $\mathbf{D}$ we calculate the corresponding normal distribution values $\mathbf{O} = [\Phi(d_{ij})]$.

For the weighting with the corresponding Poisson probabilities we generate a matrix $\mathbf{P}$ containing the probabilities for the number of jumps $\mathbf{n}$ at times $\tau_j$. For that we define the auxiliary matrices $\mathbf{N} = \mathbf{n} \otimes \mathbf{J}_{1,T}$ and $\mathbf{U} = \mathbf{J}_{m,1} \otimes (\nu \cdot \mathbf{t})$ and with the elements $P_{ij}$ of $\mathbf{P}$:

$$
P_{ij} = \frac{\exp(-\nu U_{ij}) (\nu U_{ij})^{n_{ij}}}{n_{ij}!}
$$

(C.5)

we obtain the option deltas at every $\tau_j$ as:

$$
\Delta(S, \tau) = \sum_j (\mathbf{P} \otimes \mathbf{N})
$$

(C.6)

---

\textsuperscript{15}With vectorisation the running time for optimising certainty equivalent returns with respect to pairs $(\omega^*, F^*)$ (S&P investor with $\eta = 3$ and a maturity of six month) is only 5\% of the time without vectorisation (i.e. with loops in the code).
D CPPI CERTAINTY EQUIVALENT RETURNS

Figure 1: S&P CPPI CER*s, \( \eta = 3 \) and \( T=20 \)

Figure 2: Apple CPPI CER*s, \( \eta = 3 \) and \( T=20 \)
REFERENCES


