Does Information Content of Option Prices Add Value for Asset Allocation?

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Abstract. The aim of this paper is to determine whether forward-looking option- implied returns forecasts lead to better out-of-sample portfolio performance than conventional time series models. We consider a simple two-asset setting with a risk-free asset and the S&P 500 index the risky asset with monthly allocation revisions. We carry out a comprehensive analysis with a wide range of time-series models, two risk-neutral density inference methods, two utility functions, and several performance metrics. Portfolios are compared over the period of January 1994 to April 2010.

Our main contribution is to compare the merits of implied volatility smoothing and maximum entropy risk-neutral density estimation techniques. By using bid/ask quotes in place of the closing prices, we obtain smooth probability densities using the maximum entropy principle that outperform the probability densities obtained using the implied volatility smoothing method. We also identify which moments of the option-implied probability densities contribute most to portfolio performance.

JEL Code: C14, C22, C53, G11, G13, G17

Keywords: Risk-neutral density, Real-world density, Index options, Maximum entropy, Implied volatility smoothing, Optimal portfolio

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I Introduction

The success of a portfolio manager depends critically on her ability to develop good returns forecasts. Despite advances in statistics and econometrics, formal models based on historical time series perform poorly out-of-sample and are rarely applied in practice.[1] An alternative is to add information contained in the forward-looking prices derivative contracts, such as options. In an efficient market, current option prices reflect all information available to market participants. Thus, an option-implied estimate of underlying asset's performance should be superior to estimates based solely on historical data.

In a complete market, option prices can be used to extract a risk-neutral probability density (*RND*) for the underlying asset price at maturity of the options. In theory, the *RND* can be obtained by taking the second derivative of a continuous call option price function with respect to strike (Breeden and Litzenberger, 1978). In practice, a limited number of options are traded on a given underlying and the probability density needs to be estimated given this incomplete information. Numerous parametric and non-parametric methods for inferring probability densities from the observed option prices are discussed in the literature.[2] Furthermore, a pricing kernel needs to be estimated to relate the *RND* to a subjective density that can be used for investment decision making.[3] In particular, the *RND* does not contain any information about the expected return on the underlying asset. Typically, the pricing kernel is estimated by relating the risk-neutral densities to the realised returns in a representative agent framework.

Our main aim is to determine whether forward-looking option-implied returns forecasts correspond to better out-of-sample portfolio performance than conventional backward looking time series models. We consider a simple two-asset setting with the risk-free asset represented by the 3-month Eurodollar deposit rate and the risky asset represented by the S&P 500 index. Every month, portfolios are constructed by maximising the second- and fourth-order Taylor series expansion of the certainty equivalent of the risky pay-off. We carry out a comprehensive analysis with two benchmark strategies (average historical mean-variance and moving average mean-variance), a range of time-series models (GARCH, EGARCH, and GJR-GARCH with Gaussian and t-Student innovations and up to 3 ARCH and GARCH terms), two *RND* inference methods, two utility functions (CARA and CRRA), and several performance metrics (average return, volatility, certainty equivalent return, Sharpe ratio, average turnover, and break-even transaction cost). All strategies are compared over the period of January 1994 to April 2010.

To achieve our main aim, we compare the performance of two popular non-parametric

[1] See Brandt (2009) and references therein for a discussion.

[2] Overviews of these methods are provided by Bahra (1997); Cont (1998); and Jackwerth (2004), among others.

[3] Subjective probability is also known as 'physical' probability in the literature. We use the term 'subjective' to underline that these probabilities reflect the subjective expectations of the market participants, rather than some universal physical laws.

RND estimation methods: the smoothed implied volatility approach and the maximum entropy approach. Under the smoothed implied volatility approach, the *RND* is obtained as a derivative of the continuous call price function estimated as a smooth interpolation between the observed implied volatilities.[4] In contrast, a maximum entropy aims to reflect all information contained in the available data and to maximise the uncertainty regarding any missing information without any inter- or extrapolations (Jaynes, 1982). We use the maximum entropy principle to fit the *RND*s directly to the option prices along the lines of Buchen and Kelly (1996).[5]

Finally, we assess the influence of higher moments of return distributions on portfolio performance. In particular, we consider the second-order Taylor series expansion of the certainty equivalent return, which is equivalent to the standard mean-variance criterion, and the fourth-order expansion, which takes into account skewness and kurtosis. It has long been recognised that asset returns are generally not normally distributed (Mandelbrot, 1963; Fama, 1965). However, there is no agreement in the literature whether the use of skewness and kurtosis forecasts results in better investment decisions, not least because skewness and kurtosis cannot be accurately forecasted from returns time series.[6]

The main contribution of this paper is to compare the performance of the two popular *RND* estimation techniques: implied volatility smoothing and maximum entropy estimation. Maximum entropy method is often criticised for its sensitivity to noisy data. We show that by using bid/ask quotes in place of the closing prices, one can obtain smooth probability densities that incorporate all available price information. We also compare the performance of the option-implied densities in different market environments (the protracted growth of 1994 – 2000 and the up-and-down market of 2000 – 2010). Finally, we identify which moments of the option-implied probability density contribute most to the performance of the corresponding investment strategies.

Our findings complement those of previous studies. The use of option market information for investment portfolio construction can be traced back to Manaster and Rendleman Jr (1982), who compare the stock prices implied by call option prices and the Black-Scholes formula with the actual market prices to confirm that portfolios of stocks that are undervalued relative to their theoretical Black-Scholes price outperform the portfolios of relatively overvalued stocks. Peterson and Tucker (1988) come to similar conclusions by analysing the currency market data. Bakshi, Kapadia and Madan (2003) suggest a model-free procedure for inferring the first four moments of the log-returns from option prices and establish a re-

[4]Implied volatility smoothing is pioneered by Shimko (1993) and adopted by Malz (1997); Campa, Chang and Reider (1998); Rosenberg (1998); Jackwerth (2000); Bliss and Panigirtzoglou (2002); and Rosenberg and Engle (2002), among others. The smoothing spline approach of Bliss and Panigirtzoglou (2004) has become the standard spline interpolation technique and is adopted in numerous subsequent publications.

[5]Maximum entropy approach is also adopted by Stutzer (1996); McManus (1999); Coutant, Jondeau and Rockinger (2001); Jondeau and Rockinger (2002); Brody, Buckley and Constantinou (2007); and Rompolis (2010), among others.

[6]e.g. see Chunhachinda, Dandapani, Hamid and Prakash (1997); Prakash, Chang and Pactwa (2003); Jondeau and Rockinger (2006); and Guidolin and Timmermann (2008) for a discussion.

lationship between the subjective and risk-neutral moments. Xing, Zhang and Zhao (2010) consider the implied volatility curve slope as a predictor of equity returns. In a similar vein, Rehman and Vilkov (2010) assess the predictive power of risk-neutral skewness.

The recent works of Aït-Sahalia and Brandt (2008), Jabbour, Vera and Zuluaga (2008), DeMiguel, Plyakha, Uppal and Vilkov (2010), and Kostakis, Panigirtzoglou and Skiadopoulos (2010) examine if asset return forecasts based on the information content of option prices lead to better investment portfolios than forecasts based on price time series. Aït-Sahalia and Brandt develop an inter-temporal consumption-investment model based on the state prices inferred from option prices. They derive the optimal consumption and investment policies for a representative agent with CRRA utility and find them to be state-dependent. Their results are not used to assess the out-of-sample performance of investment strategies on real-life data.

Jabbour, Vera and Zuluaga (2008) consider robust portfolios obtained by minimising the worst-case conditional Value-at-Risk of Rockafellar and Uryasev (2000) (CVaR) over all possible returns distributions that are consistent with the observed option prices. Our approach is similar in the sense that we use a maximum entropy approach to estimate the *RND*s so that we fit the observed market quotes and maximise the uncertainty criterion without making any assumptions regarding the shape and the parameters of the distribution. However, we construct our portfolios to maximise the certainty equivalent return. Unlike the minimum CVaR approach, our approach is consistent with the von Neumann – Morgenstern axioms of choice under uncertainty.

DeMiguel, Plyakha, Uppal and Vilkov (2010) consider minimum variance portfolios constructed over a sample of 561 US stocks using model-free implied volatility and skewness and the option-implied correlation measure of Buss and Vilkov (2010). DeMiguel et al. find that although option-implied volatilities and correlations do not contribute to the performance of a minimum-variance portfolio, updating the returns expectations with risk-neutral variance and skewness improves the performance. In contrast, we consider portfolio with a single risky asset and use the standard expected utility maximisation framework to derive the optimal allocations. We also use a different mapping procedure for obtaining the subjective densities.

Kostakis, Panigirtzoglou and Skiadopoulos (2010) derive the subjective probability densities using the approach of Bliss and Panigirtzoglou (2004): a smoothing spline is fitted to the implied volatilities of the traded options and differentiated twice to obtain a *RND*. The *RND*s are mapped to subjective densities using the Berkowitz (2001) statistic assuming either exponential or power utility functions for the representative agent. We adopt the same procedure for calibrating the subjective mapping procedure. However, we add to the results of Kostakis et al. by comparing the performance of smoothing spline and maximum entropy *RND* estimation techniques.

Kostakis et al. (2010) estimate the *RND*s based on the prices of options on S&P500 index futures traded on the Chicago Mercantile Exchange from March 1992 to June 2002. They construct optimal portfolios for investors with exponential and power utility functions, as well as with power utility function with regret aversion, and use several performance metrics (Sharpe ratio, certainty equivalent increase relative to the benchmark portfolio and average portfolio turnover) are to compare the *ex post* performance of subjective densities and risk-neutral densities with those of allocations based on historical distributions. In contrast, we compare the performance of option-implied forecasts with that of time-series forecasts based on GARCH, EGARCH, and GJR-GARCH models with up to three ARCH and GARCH terms and Gaussian and t-Student error distributions. We use the prices of options on S&P500 index traded on the Chicago Board of Exchange and compare the *ex post* performance of alternative forecasts over January 1994 through April 2010. Our sample covers both bullish and bearish markets, including the market meltdown of 2007 – 2008 and the subsequent rebound of 2009.

The rest of the paper is structured as follows. Section II presents the theoretical background. Section III discusses the data. Portfolio optimisation and performance measurement are discussed in Section IV. Section V outlines the forecasting techniques. Section VI summarises the results, and finally, Section VII concludes.

II Theoretical background

II.1 Risk-neutral probability density estimation

A risk-neutral density function at a certain future date is the continuous-time analogue of the state-price function in an Arrow-Debreu economy. For a continuous state-space $\Omega \subset \mathbb{R}$, the risk neutral-probability measure is defined as a continuous function $q : \Omega \to \mathbb{R}_+$, such that $\int_\Omega q(\omega)\, d\omega = 1$. By definition, in a single-period arbitrage-free economy, security price $S(t)$ equals the discounted expected payoff taken under a risk-neutral measure at the end of period T:

$$S_t = B(t,T) \cdot \int_\Omega q(\omega) S(T,\omega)\, d\omega \tag{1}$$

where $B(t,T)$ is the price of the risk-free bond yielding one at the end of the period.

A unique *RND* exists when the market is complete. Assuming that a continuum of call options on the security exists and that the option price function is twice-differentiable with respect to strike, elementary claims can be manufactured by forming butterfly spreads of call options. The unique *RND* is obtained by taking the second derivative of the call price function (Breeden and Litzenberger, 1978):

$$\frac{\partial^2 C(t,K,T)}{\partial K^2} = B(t,T) \cdot q(\omega) \tag{2}$$

where $C(t,K,T)$ is the price at time t of a European call option with strike K and expiry date T; and $q(\omega)$ is the risk-neutral probability density.

A Smoothing spline approach

In reality, option prices exist for a limited number of strikes, and a unique *RND* cannot be obtained. A candidate analytical distribution or an underlying price process can be fitted to the quoted option prices. Alternatively, an *RND* can be approximated by interpolating between the data points to obtain a continuous call price function, or equivalently, implied volatility function, and taking the derivative of this approximation.[7] Either way, the additional structure imposed on the data can distort the results of the inference.

The interpolation approach has two principal shortcomings: first, the call price (or implied volatility function) needs to be extrapolated outside of the observed strike range; second, one has to ensure that the resultant *RND* is arbitrage-free. No arbitrage conditions exist both in the strike and in the maturity dimension. In the strike dimension, the call price has to be bounded, decreasing, and convex with respect to strike. Correspondingly, the probabilities have to be non-negative and to integrate to one. In the maturity dimension, for two otherwise identical American-style options with different maturities, the longer-lived option should have the higher price.[8]

Numerous interpolation techniques have been discussed in the literature. Shimko (1993) interpolates between the implied volatilities with a continuous quadratic function of the strike within the traded strike range and assumes constant implied volatility outside the traded range, which is equivalent to appending log-normal tails to the *RND*. Campa, Chang and Reider (1998) interpolate with a cubic spline function. Neither of these approaches satisfies the no arbitrage requirements.[9] Malz (1997) interpolates with a quadratic function of the call delta, rather than of the option price. This approach satisfies the no arbitrage requirements in the strike space by construction (Brunner and Hafner, 2003). Bliss and Panigirtzoglou (2002, 2004) combine the smoothing spline approach of Campa, Chang and Reider (1998) with the implied volatility/delta mapping of Malz (1997) to derive non-parametric *RND* estimates from FTSE 100 index futures option prices. Although increasingly popular, this approach does not ensure positive probabilities by construction.

B Maximum entropy approach

The maximum entropy principle is to draw any inference based on incomplete information from the probability distribution that has the maximum entropy permitted by the available

[7]Mapping the observations from option price/strike to implied volatility/delta space using the Black-Scholes formula results in better interpolation, since the implied volatility is a more linear and smooth function of strike than the call price (Malz, 1997). Once a continuous implied volatility function is obtained, it is mapped back into a call price function using the Black-Scholes formula. This method does not require the Black-Scholes assumptions to hold; instead, the model is used to map between price/strike space and implied volatility/delta space.

[8]We only consider cross-sections of options with one month to maturity. Thus, we are not concerned with the no arbitrage requirements in the maturity dimension.

[9]Numeric procedures for ensuring non-negativity of cubic spline interpolations are discussed by Monteiro, Tütüncü and Vicente (2008), among others.

information (Jaynes, 1982). In information theory, entropy reflects the level of unpredictability in a message. Shannon (1948) suggests the following entropy measure for a probability density function $q(\omega)$ defined on continuous state-space Ω:

$$H(q) = -k \int_{\Omega} q(\omega) \ln q(\omega) \, d\omega \tag{3}$$

where k is a positive constant representing the unit of measure. Entropy reduces to zero when an event becomes certain and increases to infinity when the probability density is uniform over an infinite support.

Under the maximum entropy approach, *RND* estimation from market data is a Shannon entropy maximisation problem given the expected values of functions $\nu(\mathbf{x})$ of the risk factors:

$$\max_{q} \; H = -\int_{\mathbf{x}\in\mathcal{R}} q(\mathbf{x}) \ln q(\mathbf{x}) \, d\mathbf{x} \tag{4}$$

subject to:

$$z_i = \int_{\mathbf{x}\in\mathcal{R}} \nu_i(\mathbf{x}) q(\mathbf{x}) \, d\mathbf{x}, \quad i = 1 \ldots N \tag{5}$$

where $\mathcal{R}$ is the (finite) set of possible risk factors realisations; $q(\mathbf{x})$ is the probability density; $z_0 = 1$ and $\nu_0 = 1$ to ensure that the probabilities sum up to one; and $\forall i$, $\nu_i(\mathbf{x})$ is *any* deterministic function with expectation z_i. By construction, probabilities are always non-negative and integrate to one. The martingale property can be imposed as an additional equality constraint.

The probability function is obtained by forming and solving a Lagrangian (Jondeau and Rockinger, 2002):

$$q(\mathbf{x}) = \exp\left\{ -\sum_{i=0}^{N} \lambda_i \nu_i(\mathbf{x}) \right\} \tag{6}$$

where vector $\boldsymbol{\lambda}$ is obtained by solving the system:

$$G_i(\boldsymbol{\lambda}) = \int_{\mathbf{x}\in\mathcal{R}} \nu_i(\mathbf{x}) \exp\left\{ -\sum_{j=0}^{N} \lambda_j \nu_j(\mathbf{x}) \right\} d\mathbf{x} = z_i \tag{7}$$

Maximum entropy has long been applied in statistics and econometrics, but until recently it found little application in finance.[10] Maximum entropy approach can be used to derive probability densities given the moments of the returns distribution (see e.g. Jondeau and Rockinger, 2002; Park and Bera, 2009). However, we use the maximum entropy approach to fit *RND*s directly to cross-sections of option prices along the lines of Buchen and Kelly (1996).

[10] See Beirlant, Dudewicz, Györfi and Meulen (1997) for a comprehensive overview of maximum entropy applications in statistics. Also see Zellner (1996) and Golan, Judge and Miller (1996) for econometric applications.

II.2 From risk-neutral to subjective probabilities

The relationship between risk-neutral and subjective probabilities is derived from the optimal consumption problem of the representative agent acting in a complete market under uncertainty Constantinides (1982). The derivation below is adapted from Jackwerth (2000).[11] For a continuous state-space Ω and a single consumption good (wealth) W, the representative agent is endowed with one unit of wealth at initial time t, so that $W_t = 1$, and optimises the expected utility of the terminal wealth $W(\omega)$ at time T. Mathematically, this is a constrained optimisation problem:

$$\max_{W(\omega),\omega\in\Omega} \mathbb{E}_P[u(W)] = \int_\Omega p(\omega)\cdot u(\omega, W(\omega))\,d\omega \tag{8}$$

subject to budget constraint:

$$1 = B(t,T)\cdot \int_\Omega q(\omega)\cdot W(\omega)\,d\omega \tag{9}$$

where $\mathbb{E}_P[\cdot]$ is the expectation under the subjective probability measure; $W(\omega)$ is the end-period wealth in state ω; $u(\cdot)$ is the (state-dependent) utility of wealth consumed in state ω; and $p(\omega)$ is the subjective probability density function of the representative agent.

Let the utility be time-additive and state-independent. Then by forming a Lagrangian and differentiating with respect to consumption, we arrive to the equilibrium relationship for any ω:

$$u'(W(\omega)) = B(t,T)\cdot\lambda\cdot\frac{q(\omega)}{p(\omega)} \tag{10}$$

where λ is the Lagrange coefficient expressing the marginal utility of endowment. Provided that the wealth is invested in the market portfolio yielding return $R = \frac{W_T}{W_t}$, Eq. (10) is equivalent to:

$$u'(R) = B(t,T)\cdot\lambda\cdot\frac{q(R)}{p(R)} \tag{11}$$

This relationship leads to (Aït-Sahalia and Lo, 2000):

$$p(R) = \frac{\frac{q(R)}{u'(R)}}{\int \frac{q(R)}{u'(R)}\,dR} = m(R)\cdot q(R) \tag{12}$$

where $m(R)$ is the pricing kernel.

The equivalence between Eq. (10) and Eq. (11) holds only for agents with time-additive and state-independent utility who hold the market portfolio. By definition, the representative agent holds the entire market portfolio and that portfolio only.[12] Individual agents may have

[11] A more general dynamic specification is discussed by Aït-Sahalia and Lo (2000).

[12] Positions in the risk-free rate and in derivatives cancel out between borrowers and lenders or between

different utilities and heterogeneous views and thus hold different portfolios. In a more general case, the investor's utility does not depend solely on the market return. Also, this return is not directly observable, since the market portfolio includes non-traded assets. In empirical literature, the market portfolio is usually represented by an equity index. Further assumptions are required to link returns on individual assets R_i to the return on the market portfolio R.

By taking the second derivative of the utility function with respect to R and dividing it by Eq. (11), one derives Pratt's coefficient of local absolute risk-aversion as a function of the *RND* and the subjective density:

$$A(R) = -\frac{u''(R)}{u'(R)} = \frac{q'(R)}{q(R)} - \frac{p'(R)}{p(R)} \tag{13}$$

Thus, there exists a tripartite relationship between the investor's risk attitude, the subjective probability density, and the risk-neutral probability density.

Two streams of research address this relationship. One stream, represented by Jackwerth (2000) and Rosenberg and Engle (2002), among others, studies the aggregate risk aversion inferred from the empirical relationship between the risk-neutral densities implied by option prices and the subjective densities inferred from historical returns distributions or from time-series forecasts. The empirical risk aversion inferred under this approach does not conform to the conventional assumptions of constant average or absolute risk aversion: risk aversion coefficients are found to be neither constant nor monotonous across future wealth.

Another stream aims to obtain subjective densities from option-based risk-neutral densities under the assumption of stationary risk attitude. Bliss and Panigirtzoglou (2004) estimate the risk-aversion for a given utility function by calibrating the subjective density implied by Eq. (12) to recent returns. Liu, Shackleton, Taylor and Xu (2007) develop a closed-form transformation between risk-neutral and subjective probabilities for an *RND* that is a mixture of log-normal densities or a beta density, and a power (CRRA) utility function.

The shortcoming of these studies is that for a representative agent holding the market portfolio, among the three components: the risk attitude, the risk-neutral density, and the subjective density – only the risk-neutral density can be inferred from the data. Any conclusions concerning the subjective density or the aggregate risk aversion will depend on strong and unverifiable assumptions about the remaining unobserved quantity. Also, and somewhat paradoxically, the *RND* is derived from the option prices in a representative agent framework, whereas the representative agent is not supposed to hold any derivatives.

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the two sides of a bet. The representative agent holds the market portfolio and thus has no net positions in derivative contracts, but it is not excluded that individual agents could have positions in the risk-free asset and derivatives according to their own beliefs and utilities

III Data

Our time-series forecasts are based on daily returns calculated from the S&P500 index reported by Bloomberg and the 3-month Eurodollar deposit rate reported by the Federal Reserve System, observed in January 1971 to February 2010.

Market-based forecasts are based on end-of-day quotes on European options on S&P 500 index traded on Chicago Board of Exchange for January 1990 to February 2010. The sample contains observation and delivery dates; option types (put or call); strikes; underlying prices; last best bid and ask quotes; open interest; and daily trading volumes. Data for March 1990 to October 2000 was acquired from Market Data Express; data for November 2000 to March 2010 was provided by IVolatility.com. Options expire on the Saturday following the third Friday of the expiration month. Expiration months are the next three months, plus three additional months from the March quarterly cycle. Settlement value is calculated using the opening sales price in the primary market of each component security of S&P 500 on the last business day before the expiration date.

We compare our option-implied volatility forecasts with the Chicago Board of Exchange Volatility Index (VIX). The VIX measures the expected volatility of the S&P500 index and is constructed as the square root of a 30-day par variance swap (Demeterfi et al., 1999). The index is quoted on annualised basis. The VIX closing prices covering February 1990 to February 2010 were downloaded from CBOE Volatility Indices.

Option data filtering is carried out in several steps. Since we are concerned with one-month forecasts, only the options expiring in 22 working days are considered. Long-term options (LEAPS) are excluded from the sample. Entries with zero bid quotes, zero open interest, or zero trading volume are discarded. Finally, entries with negative gammas and calls (puts) with negative (positive) deltas are eliminated. This leaves a total of 11 672 bid/ask quotes observed over 242 dates; typically, 26 to 56 bid/ask quotes on a given date.[13] On average, quoted strikes cover a moneyness range of -50% to +15% from at-the-money, with most trades occurring within -35% to 8% interval.[14] Options are traded more actively out of the money than in the money; in particular, there is an active market for deep out-of-the-money puts.

IV Portfolio optimisation and performance measurement

We consider an institutional investor who re-balances her portfolio every month when a set of options with 22 trading days to expiry is available. By construction of the option contracts, the length of each holding period is either 21 or 26 trading days. The performance is measured over February 1994 to March 2010, with 48 months of prior option data used to

[13] A maximum of 178 traded options were recorded on 20 November 2008, compared with only 5 on 21 December 1995.

[14] We define moneyness as $M_t = (1 + R_t^f)^{T-t} S_t/K$, where R^f is the risk-free rate, S_t is the spot price, and K is the strike price.

calibrate the subjective density mapping.

Portfolio optimisation and performance comparison is based on certainty equivalent (CE) return, i.e. the minimum risk-free return that the investor is willing to accept in place of the risky alternative:

$$\mathrm{CE}(R) = u^{-1}(\mathbb{E}_P[u(R)]) \tag{14}$$

where $u^{-1}(\cdot)$ is the inverse of the utility function and $\mathbb{E}_P[\cdot]$ is the expectation under the subjective probability measure.

Unlike many other measures, a certainty equivalent is consistent with the axioms of choice under uncertainty underlying the utility theory: by construction, the investor always maximises her expected utility when choosing the portfolio with the highest certainty equivalent return. Other common risk-adjusted performance measures either rely on a simplistic representation of risk or cannot be mapped to a continuous, twice-differentiable utility function with constant sign curvature.

At each rebalancing date t, a subjective forecast of the risky return $p(R_m)$ is developed using the information available at that time. The investor maximises the certainty equivalent of the portfolio return $\mathrm{CE}(R)$ over the period $[t,\ (t+1)]$, where $R = R_f + w(R_m - R^f)$; R_m is the S&P 500 equity index return, R_f is the Eurodollar deposit rate, and w is the portfolio weight of the equity index.

We consider exponential (CARA) utility:

$$\begin{aligned} u(R) &= -\exp\{-\gamma R\} \\ \mathrm{CE}(R) &= -\frac{1}{\gamma}\ln \mathbb{E}_P\left[\exp\{-\gamma R\}\right] \end{aligned} \tag{15}$$

and power (CRRA) utility:

$$\begin{aligned} u(R) &= \frac{(1+R)^{1-\gamma}}{1-\gamma} \\ \mathrm{CE}(R) &= \left(\mathbb{E}_P\left[(1+R)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} - 1 \end{aligned} \tag{16}$$

For these utility functions, maximisation of end-of-period wealth and maximisation of the certainty equivalent of the portfolio return are equivalent, as long as the wealth is fully invested at the beginning of the period. We assume that the investor can borrow and lend at the Eurodollar deposit rate and do not constrain the allocation to the risky asset.

In order to isolate the contribution of mean, variance, and higher moments forecasts, the certainty equivalent return is estimated using fourth-order Taylor series approximation.[15] For

[15] Direct expected utility optimisation was also considered. The results were not materially different from those of the four-moment approximation.

a sufficiently smooth function $f(R)$, the expansion is:

$$\mathbb{E}\left[f(R)\right] \approx \sum_{k=0}^{4} \frac{f^{(k)}(\mu)}{k!} \mathbb{E}\left[(R-\mu)^k\right] \tag{17}$$

where $f^{(k)}(\cdot)$ denotes the k-th derivative and μ is the expected portfolio return.

Noting that he certainty equivalent in Eq. (15) is proportional to the cumulant-generating function of the subjective density taken at $-\gamma R$ and using the properties of the cumulants, we approximate the certainty equivalent return under CARA utility as:

$$\mathrm{CE}(R) \approx \mu - \frac{\gamma}{2}\sigma^2 + \frac{\gamma^2}{6}\sigma^3\xi - \frac{\gamma^3}{24}\sigma^4\kappa \tag{18}$$

The four-moment approximation of the certainty equivalent return under CRRA utility is:[16]

$$\mathrm{CE}(R) \approx (1+\mu)\Big\{1+\gamma(1-\gamma)A\Big\}^{\frac{1}{1-\gamma}} - 1 \tag{19}$$

where:

$$A = -\frac{1}{2}\Big(\frac{\sigma}{1+\mu}\Big)^2 + \frac{1+\gamma}{6}\Big(\frac{\sigma}{1+\mu}\Big)^3\xi - \frac{(1+\gamma)(2+\gamma)}{24}\Big(\frac{\sigma}{1+\mu}\Big)^4(\kappa+3)$$

and μ, σ^2, ξ, and κ are mean return, variance, skewness, and excess kurtosis of the portfolio return under the subjective probability measure.

Jondeau and Rockinger (2006), among others, demonstrate that mean–variance–skewness–kurtosis approximation can serve as an efficient alternative to the more computationally intensive direct utility optimisation. Thus, in the portfolio management context, we are more interested in the moments of the subjective distribution, rather than in the exact shape of the distribution. For a given utility function, Eq. (12) can be used to infer the moments of the subjective density from the known moments of the risk-neutral density. Bakshi, Kapadia and Madan (2003) relate the skewness of the risk-neutral density to the higher moments of the subjective density, assuming CARA utility. Zhang and Xiang (2008) use the properties of the cumulant-generating function to relate the central moments of the *RND* to the shape of the implied volatility function. In contrast to these authors, we derive the subjective mean, variance, skewness and kurtosis as functions of the risk-neutral moments and the risk aversion parameter.

For an investor with CARA utility, Eq. (12) and the properties of the cumulant-generating function can be used to obtain the relationship between subjective and risk-neutral mean, variance, skewness, and excess kurtosis:[17]

[16]See Appendix A for details.
[17]See Appendix B for details.

$$\mu_P = \mu_Q + \gamma\sigma_Q^2 + \frac{\gamma^2}{2}\xi_Q\sigma_Q^3 + \frac{\gamma^3}{6}\kappa_Q\sigma_Q^4 + \sum_{k=5}^{\infty}\frac{\gamma^{k-1}\varkappa_{k_Q}}{(k-1)!} \tag{20}$$

$$\sigma_P^2 = \sigma_Q^2 + \gamma\xi_Q\sigma_Q^3 + \frac{\gamma^2}{2}\kappa_Q\sigma_Q^4 + \sum_{k=5}^{\infty}\frac{\gamma^{k-2}\varkappa_{k_Q}}{(k-2)!} \tag{21}$$

$$\xi_P = \xi_Q + \gamma\kappa_Q\sigma_Q + \sum_{k=5}^{\infty}\frac{\gamma^{k-3}\varkappa_{k_Q}}{(k-3)!} \tag{22}$$

$$\kappa_P = \kappa_Q + \sum_{k=5}^{\infty}\frac{\gamma^{k-4}\varkappa_{k_Q}}{(k-4)!} \tag{23}$$

where subscripts Q and P denote risk-neutral and subjective densities; and $\varkappa_k$ denotes the k-th cumulant of the corresponding probability density. This result generalises the relationship obtained by Liu, Shackleton, Taylor and Xu (2007) for returns distributions that are a mixture of log-normal densities or a beta density.

Optimal portfolios are constructed at each time t by maximising the certainty equivalent over the weight of the risky asset w given the risk-free rate R_f and the moments of the S&P500 return forecast $p(R_m)$. Next, out-of-sample return on the portfolios over the period $[t,(t+1)]$ is recorded. The procedure is repeated until the last date in the sample is reached. The *ex post* certainty equivalent for the portfolios corresponding to each forecast is calculated over a series of realised returns using Eq. (14).

For each forecast, we also report Sharpe ratio, average monthly portfolio turnover, and average allocation to risky asset. Our portfolio optimisation does not account for transaction costs. To assess if our performance rankings would be robust in presence of transaction costs, we calculate the break-even proportional transaction costs as:

$$\mathrm{TrC}_{i,j} = \frac{\mathrm{CE}_i - \mathrm{CE}_j}{|\overline{\Delta w}|_i - |\overline{\Delta w}|_j} \tag{24}$$

where CE_k is the *ex post* certainty equivalent return on portfolios based on k-th forecast; and $|\overline{\Delta w}|_k$ is the corresponding average absolute change in the risky asset weight (assuming that the investor holds the risk-free asset only in the beginning of the first period).

V Forecasting methods

We compare S&P500 return forecasts based on a price time-series model and on option-based non-parametric distributions. We use two alternative techniques to infer *RND*s from option prices: the maximum entropy approach of Buchen and Kelly (1996) and the smoothing spline approach of Bliss and Panigirtzoglou (2002).

Two mean-variance optimal portfolios are used as benchmarks. The first portfolio is based on the average mean and variance over all available history at each rebalancing date. The second is based on moving averages estimated over rolling windows of 25 000 daily observations.[18]

V.1 Time series model

Assuming that the market is at least semi-strongly efficient, the conditional mean model for the risky asset return is:

$$R_t = \mu + \varepsilon_t \tag{25}$$

where ε_t is an i.i.d. process that follows a Gaussian distribution or a Student's t-distribution.

We estimate symmetric (Bollerslev, 1986), exponential (Nelson, 1991) and asymmetric (GJR) (Glosten et al., 1993) GARCH models with up to three ARCH and GARCH terms on a moving window of 25 000 daily observations. The best model is selected based on the *ex post* certainty equivalent return of the corresponding strategy with monthly rebalancing. GJR-GARCH(1,1) with normally distributed innovations outperforms the more complex specifications and is also much more stable. Thus, we use the combination of Eq. (25) and GJR-GARCH(1,1) for time-series forecasting of the S&P500 index returns.[19] The first two moments of aggregated one-month returns are calculated analytically. Skewness and kurtosis are estimated using Monte Carlo simulation with 50 000 trials.[20]

V.2 Market-based models

A Maximum entropy RND

In an efficient market, prices reflect all available information; maximum entropy *RND* incorporates this information without imposing additional structure. We construct maximum entropy *RND*s that are consistent with the market quotes of traded options and maximise the uncertainty about the missing information.

Assume a one period, single risky asset economy with no arbitrage and zero transaction costs. The market price z of a derivative contract based on the terminal price S_T of the underlying asset equals its expected present value under the risk-neutral measure:

$$z = \int \nu(S_T) q(S_T)\, dS_T \tag{26}$$

where $\nu(S_T)$ is the discounted pay-off function and $q(S_T)$ is the risk-neutral probability

[18] Exponentially weighted moving average (EWMA) calculated over the same rolling window was also considered, but for all values of the smoothing factor, EWMA forecast was inferior to a simple moving average. The results are omitted for brevity.

[19] Performance of other time-series forecasts is not reported for brevity.

[20] Analytical formulae for the first four moments of the log-return distribution generated by GJR-GARCH are derived by Alexander, Lazar and Stanescu (2010).

density of the underlying price at the end of the period. The discounted pay-off function for European option with strike K is:

$$\nu(S_T) = B(t,T) \cdot \max\left\{0, I^{C,P}(S_T - K)\right\} \tag{27}$$

where $B(t,T)$ is the deterministic discount factor, and $I^{C,P}$ is an indicator that equals 1 for calls and -1 for puts.

In real financial markets, transaction costs create arbitrage-free bounds for option prices that are reflected in the bid/ask spread. The common approach in the literature is to ignore the spreads and to use the closing prices reported by exchanges. These prices may be asynchronously recorded or generated by imperfect theoretical models and may thus allow for arbitrage. Buchen and Kelly (1996), among others, observe that maximum entropy estimates are highly sensitive to noisy price data. In applying the maximum entropy principle, we assume that the 'true' option price lies anywhere within the bid/ask spread. Thus, we mitigate the potential impact of noisy prices by allowing for more uncertainty.

The *RND* at time t is estimated over a finite and discrete support $\{\mathcal{R}\}$ of 250 equal steps spanning from 15% below the minimum strike to 15% above the maximum strike in the cross-section observed at that time.[21] Then *RND* inference is a constrained optimisation problem:

$$\hat{q}(S_T) = \arg\max_{\boldsymbol{\lambda}} \left\{ - \sum_{S_T \in \{\mathcal{R}\}} q(S_T) \ln q(S_T) \right\} \tag{28}$$

subject to

$$z_i^B \leq \sum_{S_T \in \{\mathcal{R}\}} \nu_i(S_T) \hat{q}(S_T) \leq z_i^A, \quad i = 1 \ldots N \tag{29}$$

and

$$\sum_{S_T \in \{\mathcal{R}\}} \hat{q}(S_T) = 1 \tag{30}$$

where $q(X_T)$ is an exponential function (see Eq. 6):

$$q(X_T) = \exp\left\{-\lambda_0 - \sum_{i=1}^{N} \lambda_i \nu_i(X_T)\right\} \tag{31}$$

and z_i^B and z_i^A are the closing bid and ask quotes of i-th option out of N traded contracts observed at time t. The problem is solved numerically using Newton-Raphson method.

[21] Fixed support of 1% to 200% of the spot value of the S&P500 index and discrete steps of 0.50% and 0.25% of the spot were also considered. We found that a larger support and a smaller step size have a negligible impact on the results, but slow down the computations substantially.

B Smoothing spline RND

To compare our results with those of Kostakis, Panigirtzoglou and Skiadopoulos (2010), we derive an alternative set of forecasts using the smoothing natural spline technique of Bliss and Panigirtzoglou (2004). Observations are converted from price–strike space into implied volatility–delta space using Black-Scholes formula. A spline $g(\Delta, \boldsymbol{\theta})$ is fitted to these implied volatilities to obtain a continuous function as the optimal trade-off between accuracy and smoothness:

$$\min_{\boldsymbol{\theta}} \left\{ \sum_{j=1}^{M} w_i (IV_i - \widehat{IV}(\Delta_i, \boldsymbol{\theta}))^2 + \varphi \int_{-\infty}^{+\infty} g''(\Delta, \boldsymbol{\theta})^2 \, d\Delta \right\} \tag{32}$$

where IV_i is the observed implied volatility of the i-th option in the sample, $\widehat{IV}(\Delta_i, \theta)$ is the fitted implied volatility as a function of the i-th option's delta Δ_i and spline parameters θ; and w_i is the significance weight attached to the i-th observation.

Bliss and Panigirtzoglou (2004) use options' vegas as weights and thus assign much greater weights to the actively traded at-the-money options. The trade-off between accuracy and smoothness is controlled by smoothing parameter φ. Following Bliss and Panigirtzoglou, we set φ equal to 0.99[22]. An alternative approach is to weight the observations by liquidity, measured either as relative bid/ask spread, trading volume, or open interest. A liquidity-based procedure for setting the smoothing parameter in a similar context is discussed by Smirnov and Zakharov (2003).

Following Bliss and Panigirtzoglou, observations with deltas below 0.01 or above 0.99 are discarded. To extrapolate the spline, pseudo-observations are added 15% above the maximum and 15% below the minimum strike, i.e. at the minimum and maximum values of support $\{\mathcal{R}\}$.[23] These pseudo-observations have the same implied volatility as the adjacent real observations. Thus, flat tails are appended to the implied volatility smile and are then mapped into normal tails of the *RND*.[24] Fitted implied volatilities are converted back into call prices, and the *RND* is obtained by computing the second difference of these prices with respect to strike.

C Subjective density mapping

The literature does not yet offer an entirely satisfactory set of assumptions for inferring subjective probabilities from market-implied risk-neutral densities. We adopt the approach

[22] Bliss and Panigirtzoglou (2004) find that for $\varphi = 0.9$ to 0.99, the choice of a particular value does not have a visible impact on the estimation results. We have carried out the same numeric experiment to confirm that this finding holds for data.

[23] Bliss and Panigirtzoglou (2004) add pseudo-observations 3 strike distances above the top and below the bottom strike. This procedure can generate negative strikes. We find that the pseudo-observations result in a marginal decrease in risk-neutral expected return, and increases in volatility, skewness and kurtosis. None of these effects are statistically significant.

[24] In contrast, maximum entropy *RND* has exponential tails and can accommodate for more extreme values of skewness and kurtosis.

of Bliss and Panigirtzoglou (2004) that is suitable for non-parametric densities.[25]

Consider a utility function $u(R, \gamma)$ where γ is the single parameter describing the risk attitude, for example, a constant coefficient of absolute risk aversion. Assuming that γ is constant over a rolling window L, the optimal estimate $\hat{\gamma}$ is obtained by maximising the predictive power of subjective density $\hat{p}_t$ – obtained by plugging the *RND* estimate into Eq. (11) – with respect to a sample of recent historical returns $\boldsymbol{R} = \{R_{t-L}, \ldots, R_{t-1}\}$. The predictive power of the subjective density is measured by the p-value of the Berkowitz (2001) statistic. If the subjective density matched the empirical distribution of returns over the rolling window exactly, then the cumulative probabilities:

$$h(R_j) = \int_{-\infty}^{R_j} \hat{p}_t(x)\, dx \tag{33}$$

would be uniformly distributed. But there are more powerful statistics to check normality rather than uniformity. Therefore, Berkowitz (2001) introduces the further transformation:

$$g(R_j) = \Phi^{-1}\left(\int_{-\infty}^{R_j} \hat{p}_t(x)\, dx\right),\ j = (t-L)\ldots(t-1) \tag{34}$$

where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal probability distribution. Assuming ergodicity of returns over the observation window, if the forecast $\hat{p}_t$ is correct, series $g(\boldsymbol{R})$ should be i.i.d. and distributed as $N(0, 1)$.

Berkowitz (2001) suggests a likelihood ratio test of the adequacy of the forecasting distribution $\hat{p}_t$. The likelihood function is maximised if the first two moments of the subjective density are correctly specified. The null hypothesis of i.i.d. and $N(0, 1)$ distribution of $g(\boldsymbol{r})$ is tested against the alternative of first order autocorrelation with mean and variance different from 0 and 1. The test statistic is:

$$\text{LR} = -2(L(0, 1, 0) - L(\hat{\mu}, \hat{\sigma}, \hat{\rho})) \sim \chi^2(1) \tag{35}$$

where $L(\cdot)$ is the log-likelihood function of the regression model:

$$g_j - \mu = \rho(g_{j-1} - \mu) + \varepsilon_t \tag{36}$$

Although the Berkowitz test is increasingly popular in option-implied density studies, the goodness of fit of the subjective density to the empirical returns still depends on the choice of the utility function and its parameters. For instance, consider a Gaussian *RND* with mean R_f and variance σ^2 and assume CARA utility. Using Eq. (20) - Eq. (23), one can confirm that the subjective density is also normal with mean $\mu_P = (R_f + \gamma\sigma^2)$ and the same variance σ^2. Thus, the risk aversion coefficient estimate $\hat{\gamma}$ will be chosen so that μ_P matches the average historical return over the chosen window and will fluctuate accordingly, as we verify in our

[25] An alternative mapping procedure based on statistical calibration of a beta-function is discussed by Liu, Shackleton, Taylor and Xu (2007).

results. This may appear counter-intuitive, but it is a direct consequence of the assumptions that the representative agent is always fully invested in the market portfolio and that this portfolio is represented by the S&P500 index.

VI Results

VI.1 Empirical risk aversion

Figure 1 illustrates the evolution of the aggregate risk aversion estimated over 24, 36, and 48 months rolling windows assuming CARA utility 15 and using the maximum entropy *RND* estimate. The risk aversion fluctuates widely over time, ranging from -4.35 to 18.34 for the two year calibration window. Median risk aversion coefficients of 3.83 (for 24 months window), 2.88 (for 36 months window), and 2.56 (for 48 months window) are consistent with the literature. Bliss and Panigirtzoglou (2004) estimate median aggregate risk aversion at 3.2 to 3.5 for 5-week S&P500 options assuming different utility functions. Our results are also broadly consistent with the return risk premiums per unit variance of 2.25 to 2.86 obtained by Shackleton, Taylor and Yu (2010) for one-day, one-week, and two-weeks horizons.

Risk aversion estimates are closely tied to matching window average returns (see Figure 1). In case of CARA utility, the correlation between the absolute risk aversion coefficient γ and monthly returns is 0.67, 0.61, and 0.61 for 24, 36, and 48-months calibration windows. This reveals the paradoxical property of the utility-based subjective mapping. Intuitively, under constant volatility and single risky asset assumptions one would think that growth in the equity index should be fuelled by decreasing risk-aversion. But our risk aversion estimates do the opposite: they increase during periods of persistent market growth in order to assign higher subjective probabilities to higher returns and fall during protracted declines or in response to market crashes. The alternative of choosing a constant level of risk aversion would imply a constant subjective equity premium. But we find that the portfolios based on the best possible *ex post* constant risk aversion estimate are inferior to portfolios based on time-varying risk aversion, that is, on historical average returns. This may serve as evidence that risk attitude is time-varying and cannot be adequately captured by long-term averages.

VI.2 Mean, variance, skewness, and kurtosis forecasts

Figure 2 illustrates the subjective mean return forecasts based on Berkowitz (2001) likelihood maximisation over 24-month rolling window for a CARA utility.[26] Subjective mean forecasts are nearly completely determined by the moving average return. Depending on the window size, up to 95% of variance in subjective mean forecasts is explained by changes in

[26]Moment estimates are nearly identical for CARA and CRRA utilities. For brevity, only results for CARA utility are provided. For a 22-day ahead forecast, GARCH conditional mean is identical to the unconditional historical average.

the matching moving average return. The choice between maximum entropy and spline *RND* estimation has a negligible impact on the mean return forecast compared with the choice of the rolling window size.

The evolution of risk-neutral volatility forecasts (*RND* volatility, VIX index, and ATM implied volatility) and GARCH volatility forecast is illustrated by Figures 3 and 4. All risk-neutral volatility estimates are consistently higher than tge GARCH forecast and the historical realised volatility (estimated over 10-year window), with the difference being particularly pronounced in periods of high volatility. Thus, there is a positive risk-premium in risk-neutral volatility estimates. On the other hand, risk-neutral option-implied volatility forecasts are consistently below VIX. On average, smoothing spline volatility forecast is 90% of VIX and maximum entropy volatility forecast is 94% of VIX. The difference can be attributed to the densities' tails: the smoothing spline procedure of Bliss and Panigirtzoglou (2004) appends log-normal tails to the *RND* estimate beyond the observed strikes, whereas maximum entropy *RND* tails are of exponential shape (see Eq. (6)). This allows for higher variance of the maximum entropy densities.

Skewness forecasts based on CARA utility and 24-months subjective density calibration window are presented in Figure 4. In absolute terms, typical option-implied skewness forecast is much higher than the long-term historical average. Subjective skewness of maximum entropy densities is generally greater than that of spline-based densities and is also more responsive to changes in risk-aversion. The difference between risk-neutral and subjective skewness is much greater for maximum entropy estimates than for spline estimates. Using relation (20), we can attribute these differences to the impact of the excess kurtosis of maximum entropy *RND*s. For the spline-based *RND*s, the risk-neutral and subjective skewness forecasts are nearly identical, as the excess kurtosis of these densities is close to zero.

The excess kurtosis of maximum entropy *RND*s is consistent with the long-term historical average for plausible levels of aggregate risk-aversion (Figure 5), while the smoothing spline approach fails to produce densities with plausible kurtosis. For subjective maximum entropy density based on a 24-months rolling window of returns and a CARA utility, the median excess kurtosis is 2.86, compared with the long-term historic median of 3.29. In contrast, median excess kurtosis of the corresponding smoothing spline subjective density is only 0.62. For lower values of excess kurtosis, the difference between subjective and risk-neutral estimates is negligible, which implies that in such instances the cumulants of 5th and higher orders have little significance in mapping from risk-neutral to subjective kurtosis 20. However, the difference is non-negligible for higher values of risk-neutral kurtosis.

VI.3 Optimal allocations

A Overall performance

We compare optimal portfolio allocations for investors with CARA and CRRA utilities with personal risk aversion parameter γ ranging from 2 to 10, with discrete steps of 2. Table VII illustrates the performance of several benchmarks: the 3-month Eurodollar deposit, the S&P500 index (Panel A), and optimal portfolios based on average mean and variance estimated over the whole available history (Panel B) and over a 10-year rolling window (Panel C). The S&P500 index underperforms the Eurodollar deposit over our sample period. However, the portfolios based on historical averages outperform the risk-free asset. Portfolios based on rolling window averages perform substantially better than those based on all available observation history.

First, we compare the quality of volatility forecasts by combining long-term mean based on all available history with various volatility forecasts and ignoring skewness and kurtosis (Table 2). All forecasts correspond to higher certainty equivalent returns than the historical average volatility. Our results contradict with those of DeMiguel, Plyakha, Uppal and Vilkov (2010) who find risk-neutral volatility forecasts do not result in higher certainty equivalent return of investment portfolios compared to those based on sample variance: for both utility functions, portfolios based on moving average mean and risk-neutral variance outperform those based on moving average mean and variance. However, subjective transformation results in a substantial increase in performance. In line with the findings of Kostakis, Panigirtzoglou and Skiadopoulos (2010), subjective densities based on the shortest rolling window result in better asset allocations. On the other hand, the 48-months calibration window also corresponds to better allocations than the 36-months window. The GJR-GARCH time-series model provides the best variance forecast for all calibration windows, all levels of risk-aversion, and both types of utility function. The turnover of portfolios based on GJR-GARCH forecasts is about 1.7 - 2 times higher than that of portfolios based on option-implied forecasts. However, the outperformance of the GJR-GARCH forecast is likely to persist even when transaction costs are taken into account. A proportional transaction costs of at least 0.68% (for CARA utility) or 0.79%(for CRRA utility) will be required to equate the net of transaction costs performance of the GJR-GARCH forecast with that of the best performing option-implied forecasts. As for the relative performance of the maximum entropy and smoothing spline methods, the evidence is mixed: maximum entropy estimates perform better for 24- and 48-months subjective density calibration windows, but underperform in case of 36-months window.

To assess the information content of option-implied skewness and kurtosis, we consider allocations based on the moving average mean and alternative forecasts of the higher moments (Table VII). As expected, the higher moments estimated over daily data do not contribute anything to the performance of the GJR-GARCH forecast of monthly returns.

However, higher moments have a sizeable positive impact onto the performance of option-implied densities derived with maximum entropy method. The median risk-aversion increase for maximum entropy densities is 14 bp in case of CARA utility and 20 bp in case of CRRA utility, compared with 4 bp and 7 bp increase for maximum entropy densities. In result, when higher moments are taken into account, maximum entropy estimates perform better than smoothing spline-based estimates for all values of risk-aversion, all calibration windows, and both types of the utility function. Incorporation of higher moments also results in less extreme allocations and decreases the average portfolio turnover of option-implied densities by 12.4% for CARA utility and by 17.4% for CRRA utility. Thus, higher moments can be used to mitigate extreme views on return and variances. However, option-implied forecasts still underperform the GJR-GARCH forecast. A proportional transaction cost of at least 0.51% (for CARA utility) or 0.46% (for CRRA utility) will be required to balance the net of transaction cost performance of the GJR-GARCH forecast and the top performing maximum entropy forecast.

Finally, we compare allocations based on GJR-GARCH and on option-based subjective forecasts of mean, variance, skewness, and kurtosis. Since option-based subjective mean forecasts closely follow the average return over the calibration window, we also consider strategies based on GJR-GARCH forecast of variance, skewness, and kurtosis, and 24, 36, and 48-months moving average mean. Without any exceptions, shorter moving average and shorter calibration windows correspond to better portfolio performance. Maximum entropy forecasts outperform smoothing spline forecasts even when possible transaction costs are taken into account. The degree of outperformance is significantly different from the case of rolling average mean and subjective higher moments (compare Tables VII and 5). Despite the close correlation between subjective mean forecast and the average return over the calibration window (see Figure 2), portfolios based on the combination of moving-average returns and GJR-GARCH higher moments' forecast perform very poorly. However, GJR-GARCH forecast with mean model 25 outpeforms the option-implied forecasts, and an unrealistically high proportional transaction cost of about 1.50% is required to equalise the net performance of GJR-GARCH forecasts and the least actively traded portfolios based on option-implied densities.

B Robustness checks

To confirm the robustness of our performance rankings in different market environments, we compare the performance of optimal portfolios corresponding to GJR-GARCH forecast and to subjective option-implied forecasts of mean, variance, skewness and kurtosis on two non-overlapping periods. The first period covers January 1994 to March 2000. During this period the S&P500 index was growing consistently: the annualised average return and realised volatility of daily returns over this period are 15.65% and 14.35%, respectively. The average return on the Eurodollar deposit was 5.42%. The second period covers April 2000 to

March 2010 is characterised by a negative overall trend and high volatility: the annualised average return and realised volatility of daily returns over this period are -2.05% and 21.86%, respectively. The equity index was vastly outperformed by the Eurodollar deposit, which on average yielded 3.12% over this period.

Over both periods, the smoothing spline forecasts are outperformed by the maximum entropy forecasts (with the exception of 36-months calibration window in the second period). The difference in performance persists in presence of reasonable proportional transaction costs, since both types of forecasts are characterised by similar average turnovers. Over January 1994 - March 2000, the GJR-GARCH forecast is outperformed by maximum entropy and smoothing spline forecasts based on a 24-months calibration window, as well as by the maximum entropy forecast based on 36-months calibration window. The portfolio performance over the two periods is summarised in Tables 5 and VII. In the first period, strategies based on option market forecasts outperform the strategy based on GJR-GARCH forecast by shorting the equity index during the stagnant year 1994 and taking more aggressive long positions during the protracted equity market growth of 1996 to 2000. The cumulative returns of the maximum entropy subjective forecast based on 24-months calibration window and CARA utility transformation and of the GJR-GARCH forecast are illustrated by Figure 7.

None of the portfolios outperform the Eurodollar deposit in the second period. Option-based forecasts correctly suggest shorting S&P500 index during the bearish market of 2001 to 2002, but fail to predict the start of the subsequent rally of January 2003 to July 2007. Overall, option-based forecasts with 24-months observation window outperform the GJR-GARCH forecast over years 2000 to 2008. In particular, they suggest going short on the index during the sharp equity market slide of year 2008, while GJR-GARCH forecast corresponds to a long-only strategy during the whole period. However, the option-based forecasts fail to time the market recovery in 2009, and thus the GJR-GARCH forecast ends up as the top performer over the whole period both in terms of the cumulative return and the certainty equivalent return (see Figure 8 for the evolution of cumulative returns).

One can argue that the differences in the densities estimated under the two approaches, and thus in the corresponding optimal portfolios, arise exclusively due to the impact of the exponential tails of the maximum entropy *RND*s that lie outside of the traded strike space and thus contain little or no information. In our sample, 97.3% to 99.8% of the maximum entropy *RND* probability densities lie within the traded strike range. Censoring the tails at the maximum and minimum traded strikes typically decreases skewness and kurtosis by 2% to 20% and by 38% to 59%, respectively. However, the overall pattern of maximum entropy moments and smoothing spline skewness and kurtosis is unaffected by the censoring. The censoring does not have any visible impact onto the mean and variance of the maximum entropy distributions. We also carry out several numerical experiments to assess the sensitivity of optimal allocations to errors in volatility, skewness, and kurtosis estimates. Discrepancies between the estimates of the higher moments obtained under different sets of assumptions (maximum entropy vs. smoothing spline for *RND* inference, CARA vs. CRRA utility and various lengths of rolling window for subjective density mapping) have a negligible impact onto the optimal asset allocation. For instance, in case of mean-variance optimisation based on historical averages, by inflating the historical average volatility by a factor of 1.25, average allocation to risky asset decreases by 3.15% (for $\gamma = 10$) to 15.8% (for $\gamma = 1$). If historical average mean and variance are combined with two times the risk-neutral skewness, average allocation to risky asset decreases by 1.0% (for $\gamma = 10$) to 5.0% (for $\gamma = 1$). Finally, by combining historical average mean, variance, and skewness with risk-neutral maximum entropy kurtosis and inflating the kurtosis by a factor of 5, optimal allocation to the risky asset decreases by 1.37% (for $\gamma = 10$) to 6.9% (for $\gamma = 1$). Thus, we confirm the observation of Michaud (1989) that it is the errors in mean estimates that contribute most to asset mis-allocation.

VII Conclusions

Using a simple two-asset portfolio allocation, we find that over some observation periods and for certain assumptions about the evolution of risk attitudes, option-implied forecasts result in allocations that are superior to portfolios based on time-series models or historic averages. In particular, option-implied forecasts outperform the GJR-GARCH forecast during the prolonged period of equity market growth from January 1994 to March 2000 by suggesting more aggressive long positions on the equity index. They underperform over April 2000 to April 2010 mostly because they fail to capture the rapid S&P500 index recovery in 2009. We use a utility-based transformation to map risk-neutral densities into subjective densities. This transformation forces the subjective return expectations to trail the recent realised returns, and thus subjective densities can fail to capture rapid shifts in the market. However, subjective expected return estimates lead to better portfolios than risk-neutral forecasts, time-series forecasts, and average returns observed over the subjective density calibration window.

We find that portfolios based on maximum entropy *RND* estimates typically perform better than those based on smoothing spline densities, especially when skewness and kurtosis are taken into account. Skewness and kurtosis forecasts inferred from option prices do improve the portfolio performance by mitigating the more extreme allocations implied by the mean-variance criterion. The performance improvement is more pronounced for maximum entropy densities: the smoothing spline approach fails to capture the time-varying skewness and to generate sufficiently high kurtosis consistent with the historical average. Another advantage of the maximum entropy approach is that it allows for straightforward incorporation of bid/ask spreads, which allows for a more adequate representation of the information content of option prices and mitigates the impact of noisy data.

These findings suggest several avenues for further research. A more general HARA function can be used to improve the assessment of aggregate risk attitude and the subjective probability mapping. Option-implied forecasts could be benchmarked against Markov switching models that have been demonstrated to produce better asset allocations than traditional GARCH models (Guidolin, 2010). Multi-period portfolio optimisation with transaction costs could be used to develop more realistic asset allocation strategies. The influence of higher moments inferred from option prices can be studied with single-stock option data. Finally, the *RND* inference framework could be extended to the multi-asset case using basket options data.

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Appendices

A Approximation of CRRA certainty equivalent

Consider an investor with CRRA utility:

$$U(R) = \frac{(1+R)^{1-\gamma}}{1-\gamma} \tag{37}$$

where R is the return on the investor's portfolio. By using the fourth-order Taylor series expansion 17, we obtain an approximation of the expected utility in terms of mean, variance, skewness, and excess kurtosis of the subjective returns distribution $\{\mu, \sigma^2, \xi, \kappa\}$:

$$\begin{aligned}\mathbb{E}_P[u(R)] \approx & \frac{(1+\mu)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2(1+\mu)^{\gamma+1}}\sigma^2 + \\ & + \frac{\gamma(\gamma+1)}{6(1+\mu)^{\gamma+2}}\sigma^3\xi - \frac{\gamma(\gamma+1)(\gamma+2)}{24(1+\mu)^{\gamma+3}}\sigma^4(\kappa+3)\end{aligned} \tag{38}$$

Then the certainty return equivalent of R is approximately:

$$\text{CE}(R) \approx (1+\mu)\Big\{1+\gamma(1-\gamma)A\Big\}^{\frac{1}{1-\gamma}} - 1 \tag{39}$$

where:

$$A = -\frac{1}{2}\Big(\frac{\sigma}{1+\mu}\Big)^2 + \frac{1+\gamma}{6}\Big(\frac{\sigma}{1+\mu}\Big)^3\xi - \frac{(1+\gamma)(2+\gamma)}{24}\Big(\frac{\sigma}{1+\mu}\Big)^4(\kappa+3)$$

The second-order Taylor series expansion of Eq. (39) around zero is:

$$\text{CE}(R) = \mu + \gamma(1+\mu)(A + \frac{\gamma^2}{2}A^2) \tag{40}$$

By ignoring the second-order term and re-arranging as required, we obtain:

$$\text{CE}(R) = \mu - \frac{\gamma}{2}\frac{\sigma^2}{1+\mu} - \frac{\gamma(1+\gamma)}{6}\frac{\sigma^3}{(1+\mu)^2}\xi - \frac{\gamma(1+\gamma)(2+\gamma)}{24}\frac{\sigma^4}{(1+\mu)^3}(\kappa+3) \tag{41}$$

B From risk-neutral to subjective moments

The moment-generating function of a probability density $f(R)$ is defined as:

$$\mathcal{M}(t) \equiv \int \exp\{tR\} f(R)\, dR \tag{42}$$

Consider an investor with CARA utility:

$$U(R) = -\exp(-\gamma R) \tag{43}$$

where R is the return on investor's portfolio. Then by substituting Eq. (42) into Eq. (12), we obtain the relationship between the subjective and the risk-neutral moment-generating functions:

$$\mathcal{M}_P(t) = \frac{\int \exp\{(\gamma + t)R\} q(R)\, dR}{\int \exp\{\gamma R\} q(R)\, dR} = \frac{\mathcal{M}_Q(\gamma + t)}{\mathcal{M}_Q(\gamma)} \tag{44}$$

Mean, variance, skewness, and excess kurtosis can be easily derived using the cumulant-generating function:

$$\mathcal{G}(t) \equiv \ln \mathcal{M}(t) = \sum_{k=1}^{\infty} \frac{t^k \varkappa_k}{k!} \tag{45}$$

where $\varkappa_k = \mathcal{G}^{(k)}(0)$ is the k-th cumulant of the distribution. The first five cumulants are related to mean, variance, skewness, and excess kurtosis $(\mu, \sigma^2, \xi, \kappa)$ as:

$$\begin{aligned} \varkappa_1 &= \mu \\ \varkappa_2 &= \sigma^2 \\ \varkappa_3 &= \sigma^3 \xi \\ \varkappa_4 &= \sigma^4 \kappa \\ \varkappa_5 &= \mu_5 - 10\sigma^2 \xi \end{aligned} \tag{46}$$

where μ_5 is the 5-th central moment of the distribution. By substituting Eq. (45) into Eq. (44), we obtain the subjective moment-generating function as:

$$\begin{aligned} \mathcal{G}_P(t) &= \ln \mathcal{M}_P(t) = \ln \mathcal{M}_Q(\gamma + t) - \ln \mathcal{M}_Q(\gamma) = \\ &= \mathcal{G}_Q(\gamma + t) - \mathcal{G}_Q(\gamma) \end{aligned} \tag{47}$$

where

$$\mathcal{G}_Q(\gamma + t) = \sum_{k=1}^{\infty} \frac{(\gamma + t)^k \varkappa_{k_Q}}{k!} \tag{48}$$

The relationship between subjective and risk-neutral mean, variance, skewness, and excess kurtosis is obtained by differentiating Eq. (47) at zero and re-arranging as needed:

$$\mu_P = \mu_Q + \gamma\sigma_Q^2 + \frac{\gamma^2}{2}\xi_Q\sigma_Q^3 + \frac{\gamma^3}{6}\kappa_Q\sigma_Q^4 + \sum_{k=5}^{\infty}\frac{\gamma^{k-1}\varkappa_{k_Q}}{(k-1)!} \tag{49}$$

$$\sigma_P^2 = \sigma_Q^2 + \gamma\xi_Q\sigma_Q^3 + \frac{\gamma^2}{2}\kappa_Q\sigma_Q^4 + \sum_{k=5}^{\infty}\frac{\gamma^{k-2}\varkappa_{k_Q}}{(k-2)!} \tag{50}$$

$$\xi_P = \xi_Q + \gamma\kappa_Q\sigma_Q + \sum_{k=5}^{\infty}\frac{\gamma^{k-3}\varkappa_{k_Q}}{(k-3)!} \tag{51}$$

$$\kappa_P = \kappa_Q + \gamma\Big(\frac{\mu_5}{\sigma^4} - \frac{10\xi}{\sigma^2}\Big) + \sum_{k=6}^{\infty}\frac{\gamma^{k-4}\varkappa_{k_Q}}{(k-4)!} \tag{52}$$

The last term in each expression can be dropped in empirical approximations. Bakshi, Kapadia and Madan (2003) demonstrate that the results presented in Eq. (49) - Eq. (52) can be generalised for all marginal utility functions of the type:

$$u'(R) = \int \exp\{-zR\}\nu(dz) \tag{53}$$

where $\nu(\cdot)$ is some probability measure defined on $\mathbb{R}_+$. For positive $\nu(\cdot)$, specification (53) includes all monotone utility functions. In particular, if $\nu(\cdot)$ follows a gamma-distribution, then $u'(R)$ is a HARA marginal utility function.

For some types of the utility functions and some probability densities, the relation between the risk-neutral and the subjective mean, variance, skewness and kurtosis can be derived analytically without any approximations. We consider the case of exponential (CARA) and power (CRRA) utility and normal and gamma distributions.

B.1 Exponential utility

Consider an investor whose preferences are defined by exponential (CARA) utility (Eq. (43)). By substituting the derivative of this utility function into Eq. (12), we obtain:

$$p(R) \propto \exp\{\gamma R\} \cdot q(R) \tag{54}$$

Normal distribution. Let the risk-neutral density $q(R)$ be normal with mean μ and variance σ^2:

$$q(R) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big\{-\frac{1}{2}\Big(\frac{R-\mu}{\sigma}\Big)^2\Big\} \tag{55}$$

Then the subjective density $p(R)$ is also normal with mean $\mu' = (\mu + \gamma\sigma^2)$:

$$\begin{aligned} p(R) &\propto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{1}{2}\frac{(R-\mu)^2 - 2\gamma\sigma^2 R}{\sigma^2}\Big\} \propto \\ &\propto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{1}{2}\left(\frac{R-(\mu+\gamma\sigma^2)}{\sigma}\right)^2\Big\} \end{aligned} \tag{56}$$

Thus, the subjective density is normal with mean $\mu_P = (\mu + \gamma\sigma^2)$. Variance and higher moments of the distribution remain are not affected by the transformation.

Gamma-distribution. Let the risk-neutral density $q(R)$ be a Γ-distribution with shape α and rate β:

$$q(R) = \frac{\beta^\alpha}{\Gamma(\alpha)} R^{(\alpha-1)} \exp\{-\beta R\} \tag{57}$$

Mean, variance, skewness and kurtosis of the *RND* are:

$$\mu_Q = \frac{\alpha}{\beta}; \quad \sigma_Q^2 = \frac{\alpha}{\beta^2}; \quad \xi_Q = \frac{2}{\sqrt{\alpha}}; \quad \kappa_Q = \frac{6}{\sqrt{\alpha}} \tag{58}$$

The corresponding subjective density is also a Γ-distribution with shape α and a new rate $\beta' = (\beta - \gamma)$:

$$p(R) = \frac{\beta^\alpha}{\Gamma(\alpha)} R^{(\alpha-1)} \exp\{-(\beta-\gamma)R\} \tag{59}$$

The subjective expected return and volatility are:

$$\mu_P = \mu_Q\left(1 + \frac{\gamma}{\beta-\gamma}\right) \tag{60}$$

$$\sigma_P = \sigma_Q\left(1 + \frac{\gamma}{\beta-\gamma}\right) \tag{61}$$

where indices Q and P denote estimates under risk-neutral and subjective measure, respectively. Skewness and kurtosis are unaffected by the subjective transformation, since they are independent of β. Empirically, typical values of β' are positive, and expected return and variance are equally inflated by the subjective transformation.

B.2 Power utility

Consider an investor whose preferences are defined by power (CRRA) utility (Eq. (16)). By substituting the derivative of this utility function into Eq. (12), we obtain:

$$p(R) \propto R^\gamma q(R) \tag{62}$$

Normal distribution. Let Eq. (55) describe the risk-neutral density $q(R)$. The corresponding subjective density is:

$$p(R) \propto \frac{R^\gamma}{\sqrt{2\pi\sigma^2}} \exp\Big\{ -\frac{1}{2}\Big(\frac{R-\mu}{\sigma}\Big)^2 \Big\} \tag{63}$$

Raw moments of the subjective density are related to the raw moments of the *RND* as:

$$\mathbb{E}_P\{R_i^k\} = E_Q\{R_i^{k+\gamma}\}$$

Analytical expressions for the raw moments and correspondent cumulants exist for integer values of γ. Empirical experiments demonstrate that for typical values of risk-neutral moments and γ the transformation can result in negative subjective variance. This casts some doubt onto the applicability of power utility and normal returns distribution assumption for subjective density mapping.

Log-normal distribution. Liu, Shackleton, Taylor and Xu (2007) demonstrate that for power utility function, utility-based transformation (55) of a log-normal *RND* yields a log-normal subjective density with mean $\mu_P = \mu_Q \cdot \exp\{\gamma\sigma^2\}$. Variance and higher moments of the distribution are unaffected by the transformation. The same result holds for a mixture of log-normal densities.

Gamma-distribution. Let Eq. (57) describe the risk-neutral density $q(R)$. Then the corresponding subjective density is Γ-distributed with a different shape $\alpha' = (\alpha+\gamma)$ and the same rate β:

$$q(R) \propto \frac{\beta^\alpha}{\Gamma(\alpha)} R^{(\alpha+\gamma-1)} \exp\{-\beta R\} \propto \frac{\beta^{(\alpha+\gamma)}}{\Gamma(\alpha+\gamma)} R^{(\alpha+\gamma-1)} \exp\{-\beta R\} \tag{64}$$

The transformation affects all four moments of the distribution:

$$\begin{aligned} \mu_P = \mu_Q \cdot \Big(1 + \frac{\gamma}{\alpha}\Big) \quad &; \quad \sigma_P = \sigma_Q \cdot \Big(1 + \frac{\gamma}{\alpha}\Big) \\ \xi_P = \xi_Q \cdot \sqrt{1 - \frac{\gamma}{\alpha+\gamma}} \quad &; \quad \kappa_P = \kappa_Q \cdot \sqrt{1 - \frac{\gamma}{\alpha+\gamma}} \end{aligned} \tag{65}$$

Thus, for any risk-averse representative investor, the subjective transformation increases the mean and variance of the forecasted probability density and shrinks the skewness and kurtosis towards zero.

Table 1: Performance of benchmark portfolios: January 1994 to March 2010

Panel A: Asset performance

3-month Eurodollar deposit

Risk aversion (γ)	2	4	6	8	10
Average return	4.06%	4.06%	4.06%	4.06%	4.06%
Volatility	0.55%	0.55%	0.55%	0.55%	0.55%
CER	4.06%	4.05%	4.05%	4.05%	4.05%

S&P500 index

Risk aversion (γ)	2	4	6	8	10
Average return	0.59%	0.59%	0.59%	0.59%	0.59%
Average excess return	0.16%	0.16%	0.16%	0.16%	0.16%
Volatility	17.39%	17.39%	17.39%	17.39%	17.39%
Sharpe ratio	0.92%	0.92%	0.92%	0.92%	0.92%
CER	4.00%	0.50%	-3.30%	-7.46%	-12.06%

Panel B: historical mean-variance, all available history

CARA utility

Risk aversion (γ)	2	4	6	8	10
Average return	6.17%	5.60%	5.18%	4.93%	4.76%
Average excess return	2.07%	1.53%	1.12%	0.88%	0.72%
Volatility	19.29%	9.64%	6.43%	4.83%	3.87%
Sharpe ratio	10.68%	15.71%	17.17%	17.73%	17.90%
CER	4.40%	4.23%	4.17%	4.14%	4.13%
Average turnover	4.37%	1.50%	0.93%	0.69%	0.56%
Average risky allocation	115.32%	57.66%	38.44%	28.83%	23.06%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
Average return	6.16%	5.61%	5.19%	4.93%	4.77%
Average excess return	2.06%	1.54%	1.13%	0.89%	0.73%
Volatility	19.59%	9.73%	6.47%	4.85%	3.89%
Sharpe ratio	10.50%	15.65%	17.14%	17.71%	17.88%
CER	4.25%	4.20%	4.16%	4.14%	4.12%
Average turnover	4.50%	1.52%	0.94%	0.69%	0.56%
Average risky allocation	117.15%	58.21%	38.72%	29.01%	23.19%

Panel C: historical mean-variance (moving window)

CARA utility

Risk aversion (γ)	2	4	6	8	10
Average return	6.91%	6.14%	5.57%	5.24%	5.02%
Average excess return	2.80%	2.05%	1.51%	1.18%	0.97%
Volatility	22.73%	11.39%	7.62%	5.73%	4.61%
Sharpe ratio	12.28%	17.87%	19.51%	20.16%	20.38%
CER	4.86%	4.45%	4.31%	4.24%	4.20%
Average turnover	9.85%	4.72%	3.25%	2.49%	2.02%
Average risky allocation	91.75%	45.87%	30.58%	22.94%	18.35%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
Average return	6.91%	6.15%	5.59%	5.25%	5.03%
Average excess return	2.79%	2.07%	1.52%	1.19%	0.98%
Volatility	23.19%	11.52%	7.69%	5.78%	4.64%
Sharpe ratio	12.00%	17.80%	19.50%	20.16%	20.39%
CER	4.70%	4.42%	4.30%	4.23%	4.19%
Average turnover	10.18%	4.82%	3.30%	2.52%	2.04%
Average risky allocation	93.92%	46.50%	30.90%	23.14%	18.49%

Table 2: Performance of volatility forecasts: January 1994 to March 2010

Panel A: CER

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	9.90%	6.94%	5.96%	5.48%	5.19%
RND (maxent)	6.03%	5.04%	4.70%	4.53%	4.43%
RND (spline)	5.34%	4.69%	4.47%	4.36%	4.30%
24m subjective density (maxent)	7.57%	5.79%	5.21%	4.91%	4.73%
24m subjective density (spline)	6.07%	5.05%	4.71%	4.54%	4.44%
36m subjective density (maxent)	5.55%	4.79%	4.54%	4.41%	4.33%
36m subjective density (spline)	5.78%	4.91%	4.62%	4.47%	4.38%
48m subjective density (maxent)	6.01%	5.02%	4.69%	4.53%	4.43%
48m subjective density (spline)	5.65%	4.83%	4.56%	4.42%	4.34%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	9.98%	6.97%	5.98%	5.49%	5.20%
RND (maxent)	5.84%	4.99%	4.69%	4.53%	4.43%
RND (spline)	5.09%	4.64%	4.45%	4.35%	4.29%
24m subjective density (maxent)	6.97%	5.59%	5.09%	4.83%	4.67%
24m subjective density (spline)	5.90%	5.03%	4.71%	4.54%	4.44%
36m subjective density (maxent)	5.37%	4.80%	4.56%	4.44%	4.36%
36m subjective density (spline)	5.63%	4.89%	4.62%	4.47%	4.39%
48m subjective density (maxent)	5.86%	5.06%	4.74%	4.57%	4.47%
48m subjective density (spline)	5.23%	4.70%	4.49%	4.38%	4.31%

Panel B: Sharpe ratio

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	30.10%	36.63%	38.61%	39.41%	39.71%
RND (maxent)	18.13%	22.88%	24.17%	24.56%	24.56%
RND (spline)	14.98%	20.11%	21.57%	22.09%	22.20%
24m subjective density (maxent)	25.09%	30.31%	31.70%	32.13%	32.12%
24m subjective density (spline)	18.24%	23.24%	24.62%	25.08%	25.12%
36m subjective density (maxent)	16.17%	21.60%	23.15%	23.71%	23.85%
36m subjective density (spline)	16.57%	21.52%	22.90%	23.35%	23.41%
48m subjective density (maxent)	18.26%	23.74%	25.27%	25.80%	25.90%
48m subjective density (spline)	16.27%	21.10%	22.35%	22.70%	22.64%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	29.98%	36.46%	38.54%	39.38%	39.70%
RND (maxent)	17.36%	22.61%	24.03%	24.48%	24.51%
RND (spline)	14.20%	19.84%	21.42%	22.00%	22.15%
24m subjective density (maxent)	22.91%	28.96%	30.58%	31.12%	31.20%
24m subjective density (spline)	17.59%	23.13%	24.65%	25.17%	25.25%
36m subjective density (maxent)	15.76%	21.92%	23.64%	24.28%	24.46%
36m subjective density (spline)	16.06%	21.48%	22.98%	23.48%	23.56%
48m subjective density (maxent)	18.35%	24.49%	26.16%	26.75%	26.88%
48m subjective density (spline)	14.46%	19.90%	21.41%	21.94%	22.05%

Table 2 (cont.): Performance of volatility forecasts: January 1994 to March 2010 **Panel C: Average monthly turnover**

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	75.09%	40.04%	27.22%	20.63%	16.60%
RND (maxent)	35.80%	18.62%	12.61%	9.54%	7.67%
RND (spline)	38.95%	20.34%	13.83%	10.48%	8.44%
24m subjective density (maxent)	46.56%	23.95%	16.16%	12.20%	9.80%
24m subjective density (spline)	41.68%	22.08%	15.03%	11.39%	9.17%
36m subjective density (maxent)	46.50%	24.15%	16.36%	12.38%	9.96%
36m subjective density (spline)	40.83%	21.39%	14.52%	10.99%	8.84%
48m subjective density (maxent)	47.78%	24.12%	16.19%	12.20%	9.80%
48m subjective density (spline)	41.39%	21.17%	14.14%	10.56%	8.38%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	80.60%	41.67%	27.99%	21.08%	16.90%
RND (maxent)	37.55%	19.10%	12.84%	9.68%	7.76%
RND (spline)	40.98%	20.90%	14.09%	10.64%	8.54%
24m subjective density (maxent)	48.82%	24.52%	16.41%	12.34%	9.88%
24m subjective density (spline)	44.03%	22.78%	15.38%	11.61%	9.32%
36m subjective density (maxent)	47.79%	24.16%	16.22%	12.22%	9.81%
36m subjective density (spline)	42.38%	21.71%	14.62%	11.02%	8.85%
48m subjective density (maxent)	51.12%	25.07%	16.63%	12.48%	10.00%
48m subjective density (spline)	43.32%	21.96%	14.74%	11.10%	8.90%

Panel D: Average risky allocation

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	249.45%	124.89%	83.26%	62.45%	49.96%
RND (maxent)	160.57%	80.29%	53.52%	40.14%	32.11%
RND (spline)	172.35%	86.17%	57.45%	43.09%	34.47%
24m subjective density (maxent)	183.34%	91.67%	61.11%	45.84%	36.67%
24m subjective density (spline)	175.20%	87.60%	58.40%	43.80%	35.04%
36m subjective density (maxent)	181.32%	90.66%	60.44%	45.33%	36.26%
36m subjective density (spline)	171.81%	85.91%	57.27%	42.95%	34.36%
48m subjective density (maxent)	178.47%	89.24%	59.49%	44.62%	35.69%
48m subjective density (spline)	167.89%	84.07%	56.12%	42.15%	33.77%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	259.51%	127.74%	84.61%	63.26%	50.51%
RND (maxent)	165.00%	81.51%	54.13%	40.51%	32.37%
RND (spline)	177.37%	87.55%	58.12%	43.50%	34.76%
24m subjective density (maxent)	192.94%	95.05%	63.06%	47.18%	37.69%
24m subjective density (spline)	181.27%	89.45%	59.38%	44.44%	35.50%
36m subjective density (maxent)	189.64%	93.47%	62.02%	46.41%	37.07%
36m subjective density (spline)	176.79%	87.29%	57.95%	43.38%	34.66%
48m subjective density (maxent)	182.66%	90.09%	59.79%	44.74%	35.74%
48m subjective density (spline)	174.56%	86.21%	57.24%	42.85%	34.24%

Table 3: Performance of volatility, skewness, and kurtosis forecasts: January 1994 to March 2010

Panel A: CER

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	9.89%	6.93%	5.96%	5.48%	5.19%
RND (maxent)	6.46%	5.25%	4.84%	4.64%	4.52%
RND (spline)	5.50%	4.77%	4.53%	4.41%	4.33%
24m subjective density (maxent)	7.65%	5.83%	5.23%	4.93%	4.75%
24m subjective density (spline)	6.17%	5.10%	4.75%	4.57%	4.46%
36m subjective density (maxent)	6.20%	5.12%	4.76%	4.57%	4.46%
36m subjective density (spline)	5.91%	4.97%	4.66%	4.50%	4.41%
48m subjective density (maxent)	6.57%	5.30%	4.88%	4.67%	4.54%
48m subjective density (spline)	5.76%	4.89%	4.60%	4.45%	4.36%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	9.95%	6.96%	5.98%	5.49%	5.19%
RND (maxent)	6.57%	5.28%	4.86%	4.65%	4.53%
RND (spline)	5.50%	4.77%	4.53%	4.41%	4.33%
24m subjective density (maxent)	7.65%	5.78%	5.18%	4.89%	4.71%
24m subjective density (spline)	6.20%	5.12%	4.76%	4.58%	4.47%
36m subjective density (maxent)	6.64%	5.30%	4.87%	4.65%	4.53%
36m subjective density (spline)	5.94%	4.99%	4.68%	4.52%	4.42%
48m subjective density (maxent)	6.86%	5.42%	4.95%	4.72%	4.58%
48m subjective density (spline)	5.60%	4.82%	4.56%	4.43%	4.35%

Panel B: Sharpe ratio

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	30.45%	36.87%	38.73%	39.45%	39.68%
RND (maxent)	20.48%	24.60%	25.63%	25.82%	25.65%
RND (spline)	15.62%	20.39%	21.71%	22.13%	22.17%
24m subjective density (maxent)	25.88%	30.48%	31.65%	31.93%	31.79%
24m subjective density (spline)	18.80%	23.44%	24.68%	25.04%	25.01%
36m subjective density (maxent)	18.75%	23.52%	24.81%	25.20%	25.19%
36m subjective density (spline)	17.24%	21.84%	23.08%	23.44%	23.42%
48m subjective density (maxent)	20.74%	25.48%	26.73%	27.08%	27.02%
48m subjective density (spline)	16.79%	21.26%	22.38%	22.63%	22.50%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	30.56%	36.87%	38.72%	39.44%	39.68%
RND (maxent)	21.32%	24.93%	25.83%	25.97%	25.76%
RND (spline)	15.73%	20.38%	21.70%	22.13%	22.17%
24m subjective density (maxent)	26.18%	30.09%	31.10%	31.32%	31.18%
24m subjective density (spline)	19.05%	23.61%	24.86%	25.23%	25.20%
36m subjective density (maxent)	20.97%	25.15%	26.28%	26.59%	26.52%
36m subjective density (spline)	17.54%	22.03%	23.27%	23.63%	23.61%
48m subjective density (maxent)	22.45%	26.64%	27.74%	28.01%	27.91%
48m subjective density (spline)	16.03%	20.53%	21.78%	22.17%	22.18%

Table VII (cont.): Performance of volatility, skewness, and kurtosis forecasts: January 1994 to March 2010

Panel C: Average monthly turnover

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	68.97%	36.62%	24.90%	18.86%	15.18%
RND (maxent)	30.44%	15.82%	10.69%	8.07%	6.48%
RND (spline)	35.77%	18.62%	12.65%	9.58%	7.71%
24m subjective density (maxent)	40.59%	20.86%	14.08%	10.62%	8.53%
24m subjective density (spline)	38.26%	20.18%	13.72%	10.40%	8.37%
36m subjective density (maxent)	40.80%	21.08%	14.28%	10.79%	8.68%
36m subjective density (spline)	37.45%	19.63%	13.30%	10.05%	8.08%
48m subjective density (maxent)	41.21%	20.80%	13.98%	10.54%	8.46%
48m subjective density (spline)	38.31%	19.56%	13.04%	9.72%	7.72%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	67.44%	36.34%	24.81%	18.83%	15.18%
RND (maxent)	28.66%	15.36%	10.49%	7.96%	6.42%
RND (spline)	34.93%	18.46%	12.58%	9.55%	7.69%
24m subjective density (maxent)	38.73%	20.44%	13.95%	10.59%	8.54%
24m subjective density (spline)	37.39%	20.02%	13.68%	10.38%	8.37%
36m subjective density (maxent)	37.72%	19.93%	13.59%	10.31%	8.31%
36m subjective density (spline)	36.19%	19.24%	13.09%	9.92%	7.98%
48m subjective density (maxent)	39.42%	20.37%	13.79%	10.43%	8.39%
48m subjective density (spline)	36.72%	19.33%	13.12%	9.93%	7.99%

Panel D: Average risky allocation

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	237.37%	118.68%	79.12%	59.34%	47.47%
RND (maxent)	143.12%	71.56%	47.71%	35.78%	28.62%
RND (spline)	161.00%	80.50%	53.67%	40.25%	32.20%
24m subjective density (maxent)	168.19%	84.10%	56.06%	42.05%	33.64%
24m subjective density (spline)	163.63%	81.82%	54.54%	40.91%	32.73%
36m subjective density (maxent)	165.51%	82.75%	55.17%	41.38%	33.10%
36m subjective density (spline)	160.63%	80.31%	53.54%	40.16%	32.13%
48m subjective density (maxent)	161.39%	80.70%	53.80%	40.35%	32.28%
48m subjective density (spline)	157.26%	78.75%	52.58%	39.49%	31.64%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	233.79%	118.15%	79.00%	59.33%	47.50%
RND (maxent)	136.28%	69.96%	47.05%	35.44%	28.43%
RND (spline)	157.35%	79.78%	53.42%	40.15%	32.16%
24m subjective density (maxent)	164.20%	84.10%	56.51%	42.54%	34.11%
24m subjective density (spline)	160.59%	81.43%	54.53%	40.98%	32.83%
36m subjective density (maxent)	160.03%	82.03%	55.13%	41.51%	33.29%
36m subjective density (spline)	157.13%	79.65%	53.33%	40.08%	32.10%
48m subjective density (maxent)	153.39%	78.60%	52.83%	39.78%	31.90%
48m subjective density (spline)	155.22%	78.68%	52.68%	39.59%	31.71%

Table 4: Performance of mean, variance, skewness, and kurtosis forecasts: January 1994 to March 2010

Panel A: CER

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	10.48%	7.22%	6.15%	5.62%	5.30%
24m moving ave + GARCH	-4.67%	-0.44%	1.00%	1.73%	2.17%
24m subjective density (maxent)	8.23%	6.10%	5.40%	5.05%	4.83%
24m subjective density (spline)	6.96%	5.48%	4.98%	4.74%	4.59%
36m moving ave + GARCH	-2.21%	0.84%	1.88%	2.40%	2.71%
36m subjective density (maxent)	6.69%	5.35%	4.90%	4.68%	4.54%
36m subjective density (spline)	5.96%	4.99%	4.66%	4.50%	4.40%
48m moving ave + GARCH	-2.57%	0.67%	1.77%	2.32%	2.65%
48m subjective density (maxent)	3.37%	3.70%	3.81%	3.87%	3.90%
48m subjective density (spline)	2.79%	3.40%	3.60%	3.70%	3.76%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	10.49%	7.24%	6.16%	5.63%	5.31%
24m moving ave + GARCH	-4.78%	-0.47%	0.99%	1.72%	2.17%
24m subjective density (maxent)	8.67%	6.15%	5.38%	5.02%	4.80%
24m subjective density (spline)	7.06%	5.51%	5.01%	4.75%	4.60%
36m moving ave + GARCH	-2.24%	0.83%	1.87%	2.40%	2.71%
36m subjective density (maxent)	7.00%	5.51%	5.01%	4.76%	4.61%
36m subjective density (spline)	5.83%	4.95%	4.65%	4.49%	4.39%
48m moving ave + GARCH	-2.61%	0.65%	1.76%	2.31%	2.65%
48m subjective density (maxent)	3.15%	3.59%	3.73%	3.81%	3.85%
48m subjective density (spline)	2.58%	3.33%	3.57%	3.68%	3.75%

Panel B: Sharpe ratio

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	33.69%	39.26%	40.80%	41.32%	41.39%
24m moving ave + GARCH	-6.08%	3.16%	6.20%	7.67%	8.52%
24m subjective density (maxent)	27.77%	33.36%	34.91%	35.42%	35.49%
24m subjective density (spline)	22.99%	28.55%	30.12%	30.67%	30.77%
36m moving ave + GARCH	-10.74%	-3.61%	-1.23%	-0.06%	0.63%
36m subjective density (maxent)	20.07%	24.20%	25.24%	25.45%	25.29%
36m subjective density (spline)	17.13%	21.02%	22.00%	22.18%	22.01%
48m moving ave + GARCH	-14.42%	-8.15%	-6.03%	-4.94%	-4.27%
48m subjective density (maxent)	4.20%	8.29%	9.49%	9.94%	10.07%
48m subjective density (spline)	-0.22%	3.36%	4.35%	4.69%	4.75%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	33.76%	39.25%	40.80%	41.32%	41.39%
24m moving ave + GARCH	-6.01%	3.16%	6.21%	7.70%	8.56%
24m subjective density (maxent)	29.11%	33.65%	34.90%	35.28%	35.29%
24m subjective density (spline)	23.23%	28.86%	30.50%	31.10%	31.23%
36m moving ave + GARCH	-10.68%	-3.62%	-1.23%	-0.04%	0.66%
36m subjective density (maxent)	21.64%	26.06%	27.28%	27.62%	27.55%
36m subjective density (spline)	16.60%	21.02%	22.23%	22.55%	22.49%
48m moving ave + GARCH	-14.57%	-8.25%	-6.07%	-4.96%	-4.27%
48m subjective density (maxent)	3.92%	8.60%	10.08%	10.70%	10.94%
48m subjective density (spline)	-0.33%	4.08%	5.47%	6.07%	6.34%

Table VII (cont.): Performance of mean, variance, skewness, and kurtosis forecasts: January 1994 to March 2010

Panel C: Average monthly turnover

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	60.12%	31.83%	21.64%	16.38%	13.18%
24m moving ave + GARCH	107.46%	55.81%	37.79%	28.62%	23.04%
24m subjective density (maxent)	69.63%	35.17%	23.61%	17.78%	14.26%
24m subjective density (spline)	65.01%	32.89%	22.11%	16.66%	13.37%
36m moving ave + GARCH	85.98%	44.33%	29.95%	22.63%	18.19%
36m subjective density (maxent)	55.79%	28.10%	18.79%	14.12%	11.30%
36m subjective density (spline)	49.49%	24.85%	16.65%	12.52%	10.03%
48m moving ave + GARCH	78.54%	40.05%	26.91%	20.27%	16.25%
48m subjective density (maxent)	49.28%	24.95%	16.70%	12.56%	10.07%
48m subjective density (spline)	43.27%	21.78%	14.51%	11.04%	8.96%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	58.86%	31.60%	21.56%	16.36%	13.18%
24m moving ave + GARCH	106.88%	56.19%	38.02%	28.77%	23.15%
24m subjective density (maxent)	67.86%	35.19%	23.83%	18.05%	14.53%
24m subjective density (spline)	63.65%	32.55%	21.95%	16.57%	13.32%
36m moving ave + GARCH	85.22%	44.32%	29.99%	22.67%	18.23%
36m subjective density (maxent)	55.12%	28.29%	19.01%	14.31%	11.48%
36m subjective density (spline)	49.60%	25.17%	16.90%	12.74%	10.21%
48m moving ave + GARCH	77.62%	39.95%	26.92%	20.29%	16.28%
48m subjective density (maxent)	45.08%	23.71%	16.08%	12.16%	9.78%
48m subjective density (spline)	43.42%	22.38%	15.08%	11.37%	9.12%

Panel D: Average risky allocation

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	210.24%	105.12%	70.08%	52.56%	42.05%
24m moving ave + GARCH	127.49%	64.65%	43.10%	32.32%	25.86%
24m subjective density (maxent)	110.69%	55.35%	36.90%	27.67%	22.14%
24m subjective density (spline)	96.81%	48.41%	32.27%	24.20%	19.36%
36m moving ave + GARCH	108.81%	54.40%	36.27%	27.20%	21.76%
36m subjective density (maxent)	90.38%	45.19%	30.13%	22.59%	18.08%
36m subjective density (spline)	75.94%	37.97%	25.31%	18.99%	15.19%
48m moving ave + GARCH	93.87%	46.94%	31.29%	23.47%	18.77%
48m subjective density (maxent)	77.84%	38.92%	25.95%	19.46%	15.57%
48m subjective density (spline)	60.26%	30.25%	20.24%	15.24%	12.24%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	207.35%	104.70%	69.99%	52.56%	42.08%
24m moving ave + GARCH	126.16%	64.72%	43.22%	32.44%	25.96%
24m subjective density (maxent)	114.77%	59.07%	39.78%	29.99%	24.07%
24m subjective density (spline)	100.99%	51.66%	34.70%	26.12%	20.94%
36m moving ave + GARCH	107.38%	54.25%	36.28%	27.25%	21.82%
36m subjective density (maxent)	90.76%	46.79%	31.53%	23.77%	19.08%
36m subjective density (spline)	78.19%	40.00%	26.86%	20.22%	16.21%
48m moving ave + GARCH	92.07%	46.69%	31.25%	23.49%	18.81%
48m subjective density (maxent)	75.61%	38.78%	26.08%	19.65%	15.76%
48m subjective density (spline)	61.34%	31.47%	21.16%	15.93%	12.78%

Table 5: Performance of mean, variance, skewness, and kurtosis forecasts: January 1994 to March 2000

Panel A: CER

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	30.58%	17.41%	13.29%	11.29%	10.10%
24m moving ave + GARCH	30.75%	17.48%	13.34%	11.32%	10.12%
24m subjective density (maxent)	35.94%	19.82%	14.85%	12.43%	11.01%
24m subjective density (spline)	32.03%	18.07%	13.72%	11.61%	10.35%
36m moving ave + GARCH	27.93%	16.20%	12.52%	10.71%	9.64%
36m subjective density (maxent)	32.00%	18.05%	13.72%	11.60%	10.35%
36m subjective density (spline)	27.35%	15.94%	12.35%	10.59%	9.55%
48m moving ave + GARCH	26.90%	15.73%	12.21%	10.49%	9.46%
48m subjective density (maxent)	24.13%	14.45%	11.39%	9.88%	8.99%
48m subjective density (spline)	22.48%	13.66%	10.85%	9.46%	8.64%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	30.65%	17.47%	13.34%	11.32%	10.12%
24m moving ave + GARCH	30.60%	17.52%	13.37%	11.35%	10.14%
24m subjective density (maxent)	37.53%	20.69%	15.44%	12.88%	11.36%
24m subjective density (spline)	33.41%	18.80%	14.21%	11.97%	10.65%
36m moving ave + GARCH	27.99%	16.26%	12.55%	10.74%	9.67%
36m subjective density (maxent)	33.70%	19.00%	14.36%	12.08%	10.74%
36m subjective density (spline)	29.04%	16.83%	12.95%	11.04%	9.91%
48m moving ave + GARCH	26.87%	15.76%	12.23%	10.51%	9.48%
48m subjective density (maxent)	26.42%	15.62%	12.16%	10.46%	9.45%
48m subjective density (spline)	23.95%	14.46%	11.41%	9.90%	9.01%

Panel B: Sharpe ratio

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	92.90%	93.61%	92.81%	91.53%	89.95%
24m moving ave + GARCH	68.91%	76.05%	77.59%	77.95%	77.83%
24m subjective density (maxent)	103.03%	103.61%	102.80%	101.60%	100.14%
24m subjective density (spline)	97.65%	98.38%	97.59%	96.32%	94.74%
36m moving ave + GARCH	66.77%	72.53%	73.69%	73.80%	73.44%
36m subjective density (maxent)	104.62%	104.47%	103.18%	101.44%	99.39%
36m subjective density (spline)	95.54%	95.91%	94.75%	93.02%	90.92%
48m moving ave + GARCH	72.09%	76.50%	77.10%	76.79%	76.05%
48m subjective density (maxent)	86.51%	87.45%	86.41%	84.69%	82.55%
48m subjective density (spline)	88.19%	88.11%	86.27%	83.76%	80.86%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	93.41%	93.82%	92.96%	91.65%	90.06%
24m moving ave + GARCH	68.16%	75.73%	77.39%	77.81%	77.73%
24m subjective density (maxent)	104.20%	104.74%	104.00%	102.91%	101.60%
24m subjective density (spline)	98.98%	99.70%	98.99%	97.82%	96.37%
36m moving ave + GARCH	66.92%	72.54%	73.70%	73.81%	73.45%
36m subjective density (maxent)	106.06%	105.88%	104.72%	103.18%	101.37%
36m subjective density (spline)	98.32%	98.62%	97.56%	95.97%	94.05%
48m moving ave + GARCH	72.45%	76.59%	77.15%	76.83%	76.09%
48m subjective density (maxent)	92.55%	92.81%	91.66%	89.99%	87.98%
48m subjective density (spline)	90.45%	90.72%	89.42%	87.49%	85.16%

Table 5 (cont.): Performance of mean, variance, skewness, and kurtosis forecasts: January 1994 to March 2000

Panel C: Average monthly turnover

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	76.87%	39.66%	26.76%	20.19%	16.21%
24m moving ave + GARCH	137.12%	73.64%	50.47%	38.46%	31.06%
24m subjective density (maxent)	86.04%	43.18%	28.90%	21.72%	17.40%
24m subjective density (spline)	72.34%	35.75%	23.98%	18.06%	14.49%
36m moving ave + GARCH	121.60%	65.04%	44.51%	33.84%	27.29%
36m subjective density (maxent)	64.40%	32.61%	21.84%	16.42%	13.15%
36m subjective density (spline)	57.85%	29.11%	19.55%	14.72%	11.80%
48m moving ave + GARCH	111.51%	58.59%	39.79%	30.12%	24.23%
48m subjective density (maxent)	60.00%	30.55%	20.50%	15.43%	12.38%
48m subjective density (spline)	56.02%	27.83%	18.40%	14.18%	11.63%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	75.31%	39.41%	26.69%	20.18%	16.22%
24m moving ave + GARCH	137.27%	74.83%	51.09%	38.85%	31.33%
24m subjective density (maxent)	85.52%	43.26%	29.07%	21.89%	17.56%
24m subjective density (spline)	70.13%	35.03%	23.55%	17.79%	14.30%
36m moving ave + GARCH	120.81%	65.13%	44.63%	33.95%	27.38%
36m subjective density (maxent)	62.26%	32.21%	21.74%	16.40%	13.17%
36m subjective density (spline)	58.02%	29.48%	19.84%	14.98%	12.02%
48m moving ave + GARCH	109.97%	58.39%	39.77%	30.15%	24.27%
48m subjective density (maxent)	56.35%	29.46%	19.95%	15.09%	12.13%
48m subjective density (spline)	55.18%	28.68%	19.39%	14.64%	11.77%

Panel D: Average risky allocation

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	232.51%	116.26%	77.50%	58.13%	46.50%
24m moving ave + GARCH	337.17%	170.92%	113.95%	85.46%	68.37%
24m subjective density (maxent)	222.13%	111.06%	74.04%	55.53%	44.43%
24m subjective density (spline)	196.34%	98.17%	65.45%	49.08%	39.27%
36m moving ave + GARCH	306.01%	153.01%	102.00%	76.50%	61.20%
36m subjective density (maxent)	179.75%	89.87%	59.92%	44.94%	35.95%
36m subjective density (spline)	162.32%	81.16%	54.11%	40.58%	32.46%
48m moving ave + GARCH	282.55%	141.27%	94.18%	70.64%	56.51%
48m subjective density (maxent)	170.30%	85.15%	56.77%	42.58%	34.06%
48m subjective density (spline)	146.20%	73.41%	49.14%	37.01%	29.73%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	229.70%	115.95%	77.50%	58.20%	46.59%
24m moving ave + GARCH	334.66%	171.40%	114.37%	85.80%	68.65%
24m subjective density (maxent)	232.43%	118.83%	79.79%	60.05%	48.14%
24m subjective density (spline)	205.21%	104.53%	70.10%	52.73%	42.26%
36m moving ave + GARCH	303.02%	152.81%	102.11%	76.66%	61.36%
36m subjective density (maxent)	189.61%	97.01%	65.15%	49.04%	39.31%
36m subjective density (spline)	169.69%	86.41%	57.94%	43.58%	34.92%
48m moving ave + GARCH	278.93%	140.89%	94.19%	70.73%	56.62%
48m subjective density (maxent)	176.52%	90.30%	60.66%	45.66%	36.61%
48m subjective density (spline)	154.63%	78.64%	52.71%	39.64%	31.76%

Table 6: Performance of mean, variance, skewness, and kurtosis forecasts: April 2000 to March 2010

Panel A: CER

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	-0.37%	1.40%	2.00%	2.30%	2.48%
24m moving ave + GARCH	-21.55%	-10.00%	-5.81%	-3.66%	-2.34%
24m subjective density (maxent)	-5.95%	-1.50%	0.02%	0.79%	1.26%
24m subjective density (spline)	-6.08%	-1.57%	-0.02%	0.76%	1.23%
36m moving ave + GARCH	-17.18%	-7.53%	-4.10%	-2.34%	-1.27%
36m subjective density (maxent)	-6.44%	-1.75%	-0.14%	0.67%	1.16%
36m subjective density (spline)	-5.43%	-1.22%	0.21%	0.94%	1.37%
48m moving ave + GARCH	-17.25%	-7.56%	-4.11%	-2.35%	-1.27%
48m subjective density (maxent)	-7.70%	-2.40%	-0.58%	0.35%	0.90%
48m subjective density (spline)	-7.80%	-2.46%	-0.62%	0.32%	0.88%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	-0.38%	1.39%	1.99%	2.29%	2.47%
24m moving ave + GARCH	-21.64%	-10.06%	-5.86%	-3.69%	-2.36%
24m subjective density (maxent)	-6.00%	-1.86%	-0.30%	0.52%	1.02%
24m subjective density (spline)	-6.53%	-1.87%	-0.24%	0.59%	1.09%
36m moving ave + GARCH	-17.23%	-7.58%	-4.13%	-2.36%	-1.29%
36m subjective density (maxent)	-6.72%	-1.97%	-0.30%	0.55%	1.06%
36m subjective density (spline)	-6.37%	-1.72%	-0.12%	0.68%	1.17%
48m moving ave + GARCH	-17.30%	-7.61%	-4.14%	-2.37%	-1.29%
48m subjective density (maxent)	-9.04%	-3.17%	-1.11%	-0.06%	0.58%
48m subjective density (spline)	-8.76%	-2.97%	-0.97%	0.05%	0.67%

Panel B: Sharpe ratio

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	-0.42%	6.01%	8.13%	9.18%	9.78%
24m moving ave + GARCH	-54.56%	-48.86%	-46.61%	-45.33%	-44.42%
24m subjective density (maxent)	-22.18%	-16.89%	-15.01%	-14.01%	-13.34%
24m subjective density (spline)	-22.38%	-16.94%	-15.01%	-13.98%	-13.30%
36m moving ave + GARCH	-72.59%	-69.38%	-67.88%	-66.77%	-65.75%
36m subjective density (maxent)	-55.17%	-52.33%	-50.84%	-49.52%	-48.18%
36m subjective density (spline)	-49.54%	-46.55%	-45.03%	-43.74%	-42.46%
48m moving ave + GARCH	-80.86%	-78.80%	-77.65%	-76.65%	-75.61%
48m subjective density (maxent)	-60.85%	-58.40%	-57.10%	-55.91%	-54.62%
48m subjective density (spline)	-69.43%	-67.37%	-65.99%	-64.52%	-62.85%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	-0.61%	5.84%	7.99%	9.05%	9.66%
24m moving ave + GARCH	-54.43%	-48.82%	-46.59%	-45.30%	-44.40%
24m subjective density (maxent)	-24.32%	-20.68%	-19.38%	-18.65%	-18.13%
24m subjective density (spline)	-24.96%	-20.09%	-18.33%	-17.36%	-16.70%
36m moving ave + GARCH	-72.38%	-69.31%	-67.84%	-66.74%	-65.73%
36m subjective density (maxent)	-54.71%	-51.78%	-50.30%	-49.08%	-47.86%
36m subjective density (spline)	-54.66%	-51.95%	-50.51%	-49.24%	-47.93%
48m moving ave + GARCH	-81.00%	-78.87%	-77.69%	-76.67%	-75.63%
48m subjective density (maxent)	-63.98%	-60.58%	-59.00%	-57.77%	-56.58%
48m subjective density (spline)	-70.42%	-68.16%	-66.86%	-65.56%	-64.11%

Table VII (cont.): Performance of mean, variance, skewness, and kurtosis forecasts: April 2000 to March 2010

Panel C: Average monthly turnover

CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	49.91%	27.08%	18.53%	14.08%	11.35%
24m moving ave + GARCH	89.34%	44.88%	30.01%	22.58%	18.11%
24m subjective density (maxent)	59.50%	30.21%	20.34%	15.34%	12.31%
24m subjective density (spline)	60.48%	31.12%	20.95%	15.79%	12.67%
36m moving ave + GARCH	63.90%	31.47%	20.89%	15.65%	12.52%
36m subjective density (maxent)	50.21%	25.14%	16.78%	12.60%	10.08%
36m subjective density (spline)	43.94%	21.98%	14.69%	11.03%	8.83%
48m moving ave + GARCH	58.04%	28.51%	18.89%	14.13%	11.28%
48m subjective density (maxent)	42.53%	21.38%	14.28%	10.72%	8.59%
48m subjective density (spline)	35.11%	17.86%	11.98%	9.01%	7.22%

CRRA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	48.83%	26.85%	18.45%	14.05%	11.34%
24m moving ave + GARCH	88.29%	44.75%	29.99%	22.59%	18.13%
24m subjective density (maxent)	56.93%	30.20%	20.58%	15.67%	12.65%
24m subjective density (spline)	59.65%	31.01%	20.95%	15.82%	12.70%
36m moving ave + GARCH	63.14%	31.38%	20.88%	15.66%	12.54%
36m subjective density (maxent)	50.49%	25.70%	17.20%	12.92%	10.35%
36m subjective density (spline)	44.03%	22.28%	14.92%	11.22%	8.99%
48m moving ave + GARCH	57.52%	28.47%	18.91%	14.15%	11.30%
48m subjective density (maxent)	37.96%	20.04%	13.60%	10.29%	8.28%
48m subjective density (spline)	35.89%	18.31%	12.28%	9.25%	7.41%

Panel D: Average risky allocation

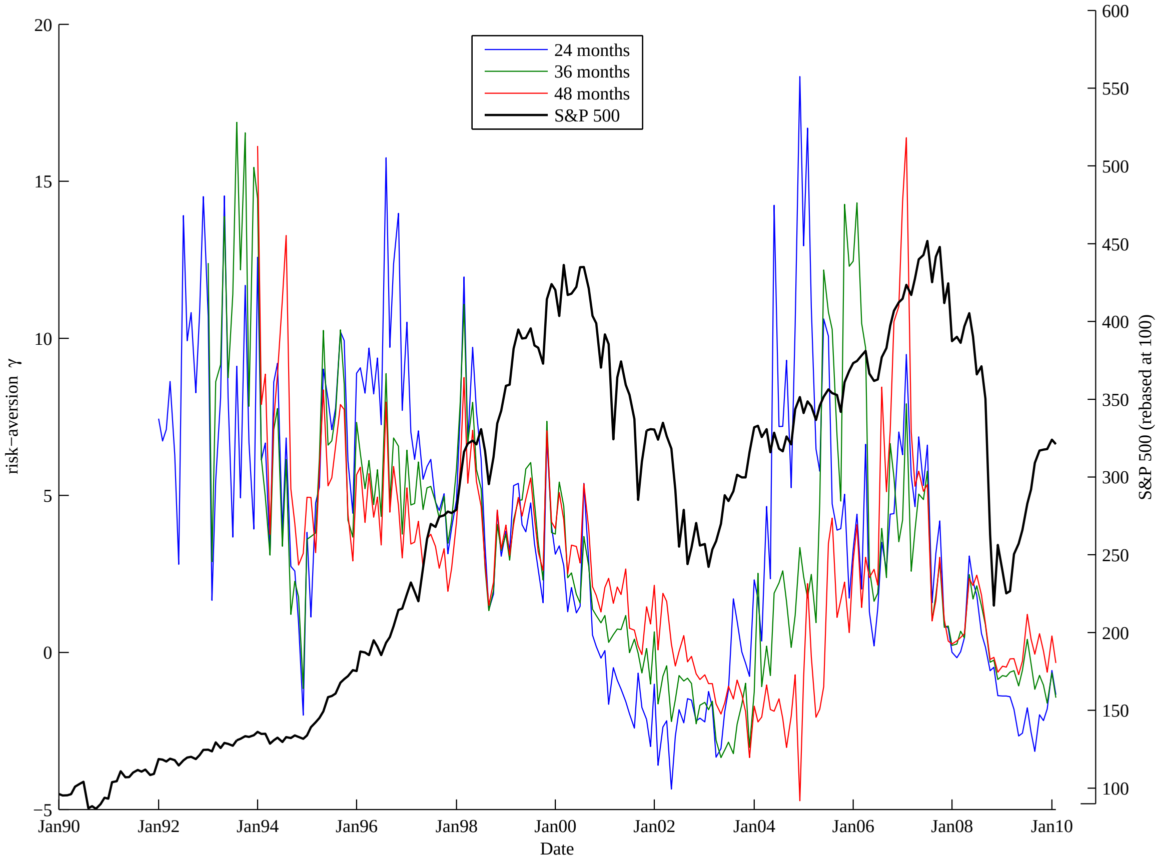
CARA utility

Risk aversion (γ)	2	4	6	8	10
GARCH	196.19%	98.10%	65.40%	49.05%	39.24%
24m moving ave + GARCH	-4.73%	-2.37%	-1.58%	-1.18%	-0.95%
24m subjective density (maxent)	40.43%	20.21%	13.48%	10.11%	8.09%
24m subjective density (spline)	34.06%	17.03%	11.35%	8.51%	6.81%
36m moving ave + GARCH	-15.54%	-7.77%	-5.18%	-3.89%	-3.11%
36m subjective density (maxent)	34.03%	17.01%	11.34%	8.51%	6.81%
36m subjective density (spline)	21.48%	10.74%	7.16%	5.37%	4.30%
48m moving ave + GARCH	-25.09%	-12.55%	-8.36%	-6.27%	-5.02%
48m subjective density (maxent)	19.54%	9.77%	6.51%	4.89%	3.91%
48m subjective density (spline)	6.07%	3.03%	2.02%	1.52%	1.21%

CRRA utility

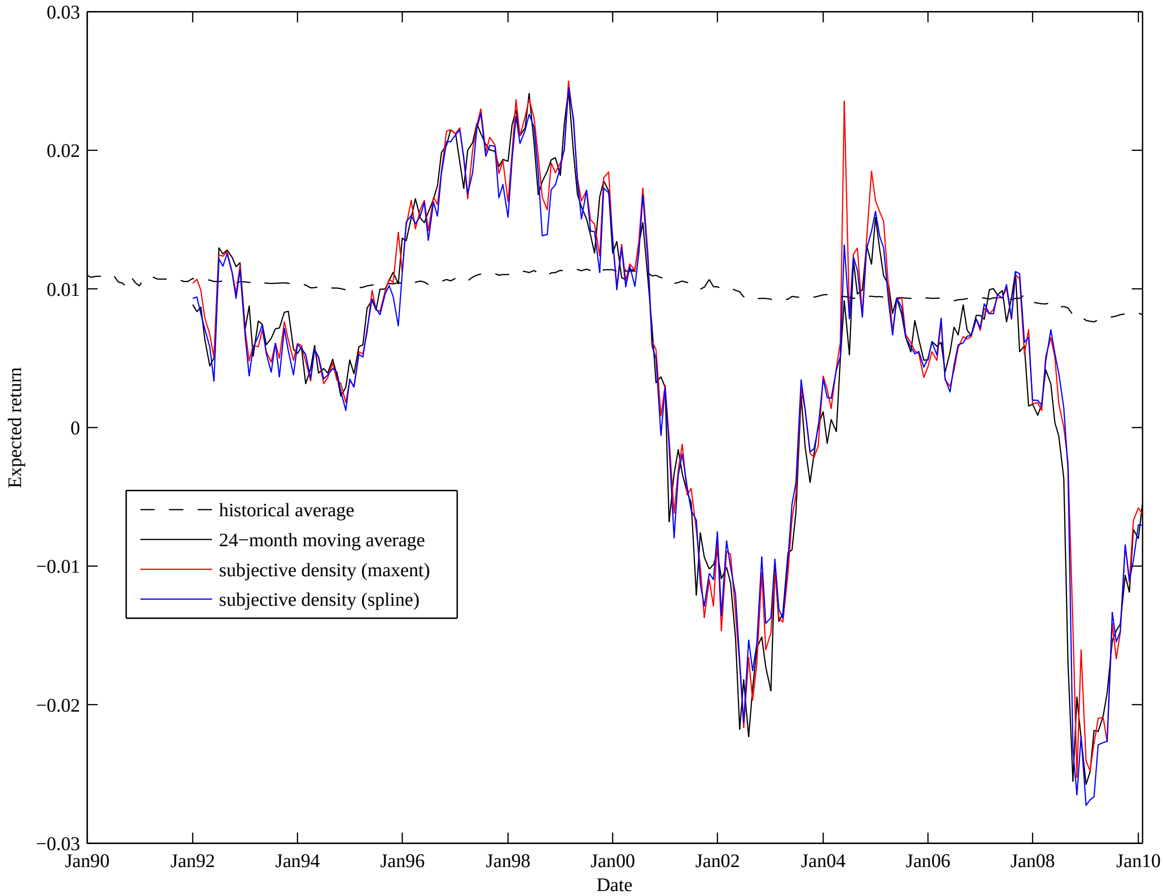
Risk aversion (γ)	2	4	6	8	10
GARCH	193.26%	97.61%	65.25%	49.00%	39.23%
24m moving ave + GARCH	-5.31%	-2.54%	-1.65%	-1.21%	-0.96%
24m subjective density (maxent)	40.57%	21.39%	14.55%	11.03%	8.88%
24m subjective density (spline)	35.27%	18.32%	12.38%	9.34%	7.50%
36m moving ave + GARCH	-15.99%	-7.89%	-5.22%	-3.90%	-3.11%
36m subjective density (maxent)	28.43%	15.13%	10.32%	7.83%	6.31%
36m subjective density (spline)	20.49%	10.73%	7.27%	5.49%	4.41%
48m moving ave + GARCH	-25.76%	-12.71%	-8.43%	-6.30%	-5.03%
48m subjective density (maxent)	11.99%	6.29%	4.27%	3.24%	2.61%
48m subjective density (spline)	2.52%	1.73%	1.26%	0.98%	0.81%

Figure 1: Risk aversion estimates and cumulative return on S&P 500



Absolute risk aversion coefficients for CARA utility function. Risk aversion estimates are nearly identical for CARA and CRRA utility.

Figure 2: Expected returns



Subjective estimates are based on Berkowitz (2001) likelihood maximisation over 24-month rolling window and CARA utility.

Figure 3: Volatility forecasts

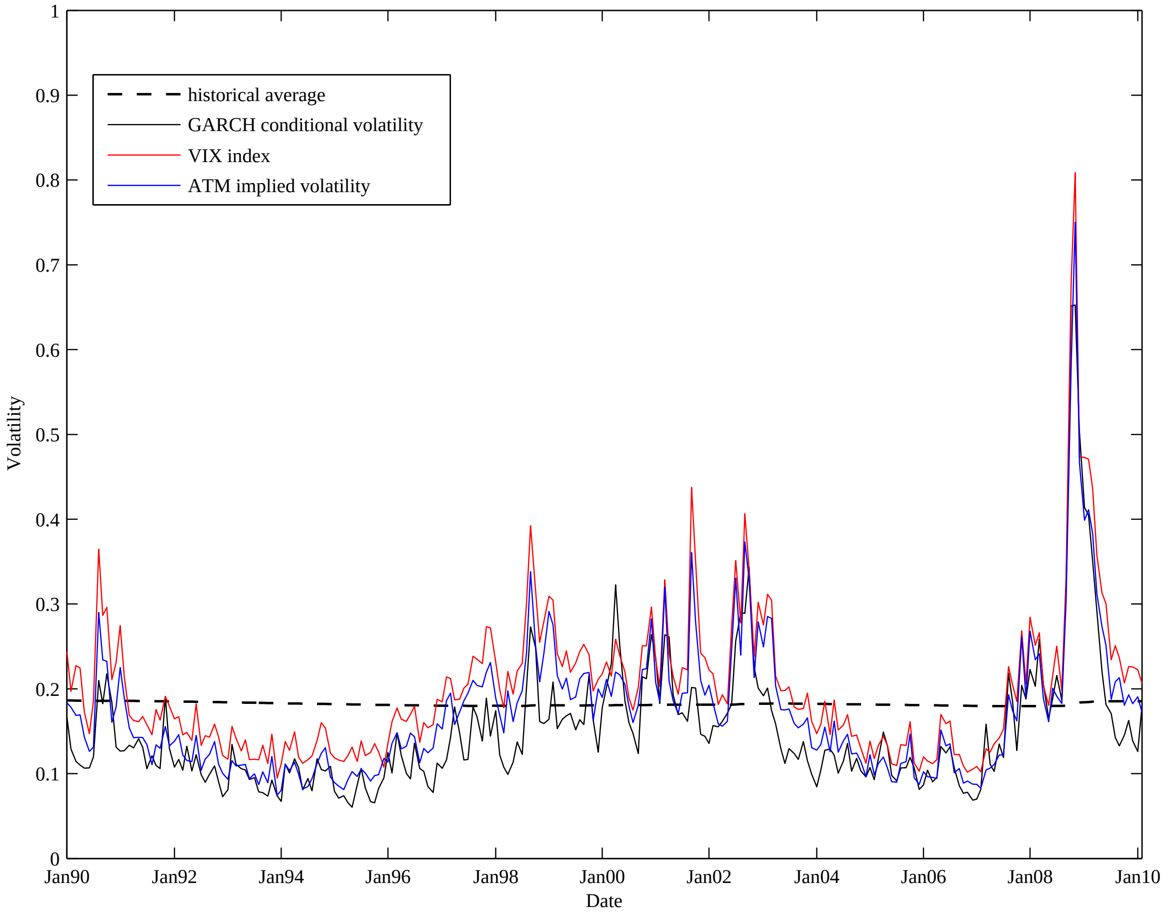


Figure 4: Volatilities of option-implied *RND*s relative to VIX.

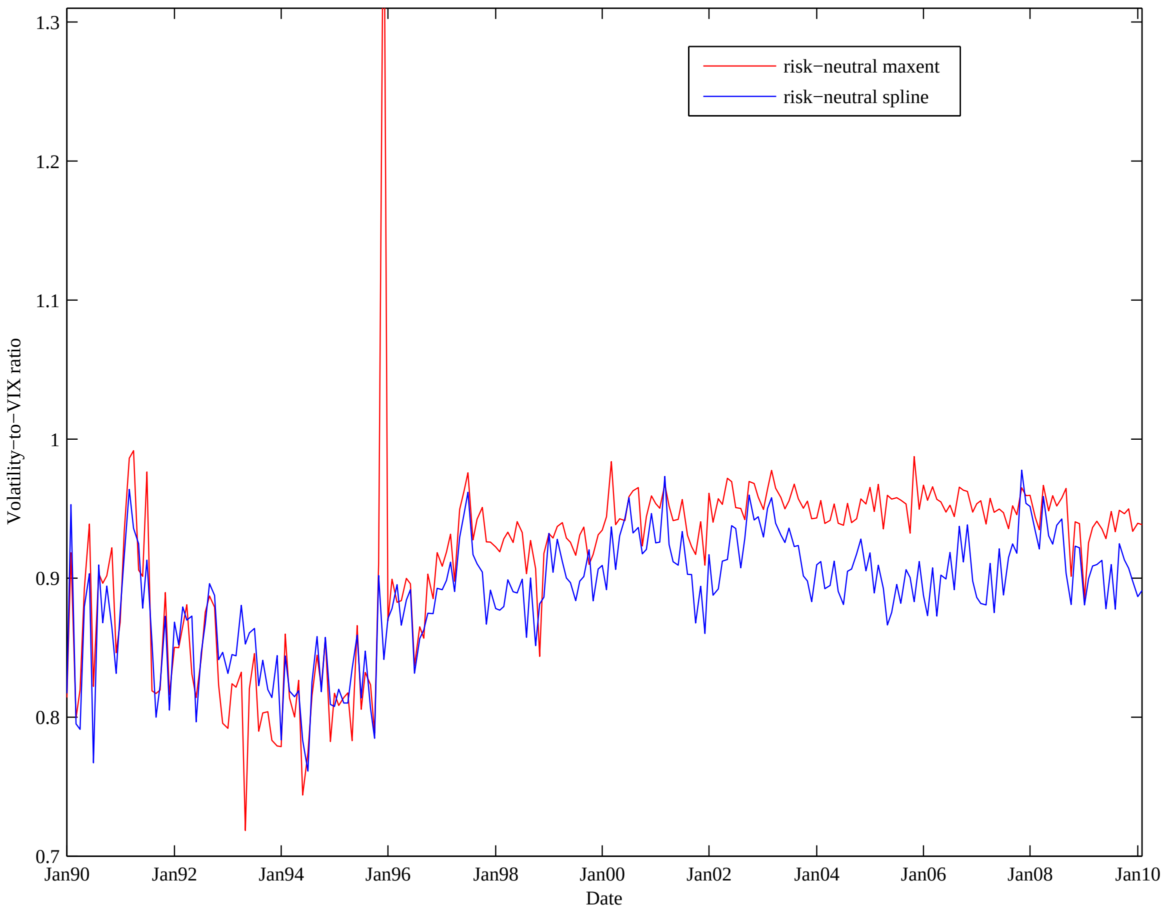
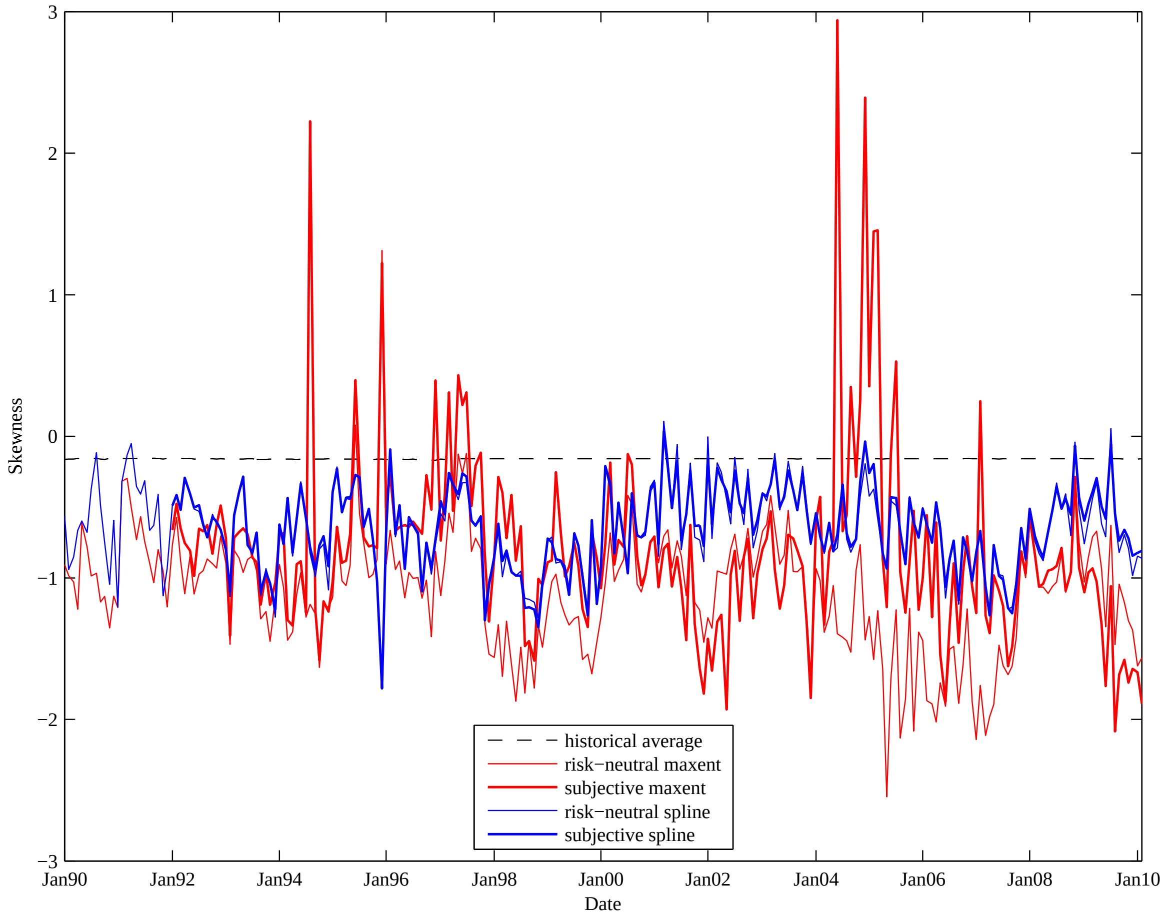
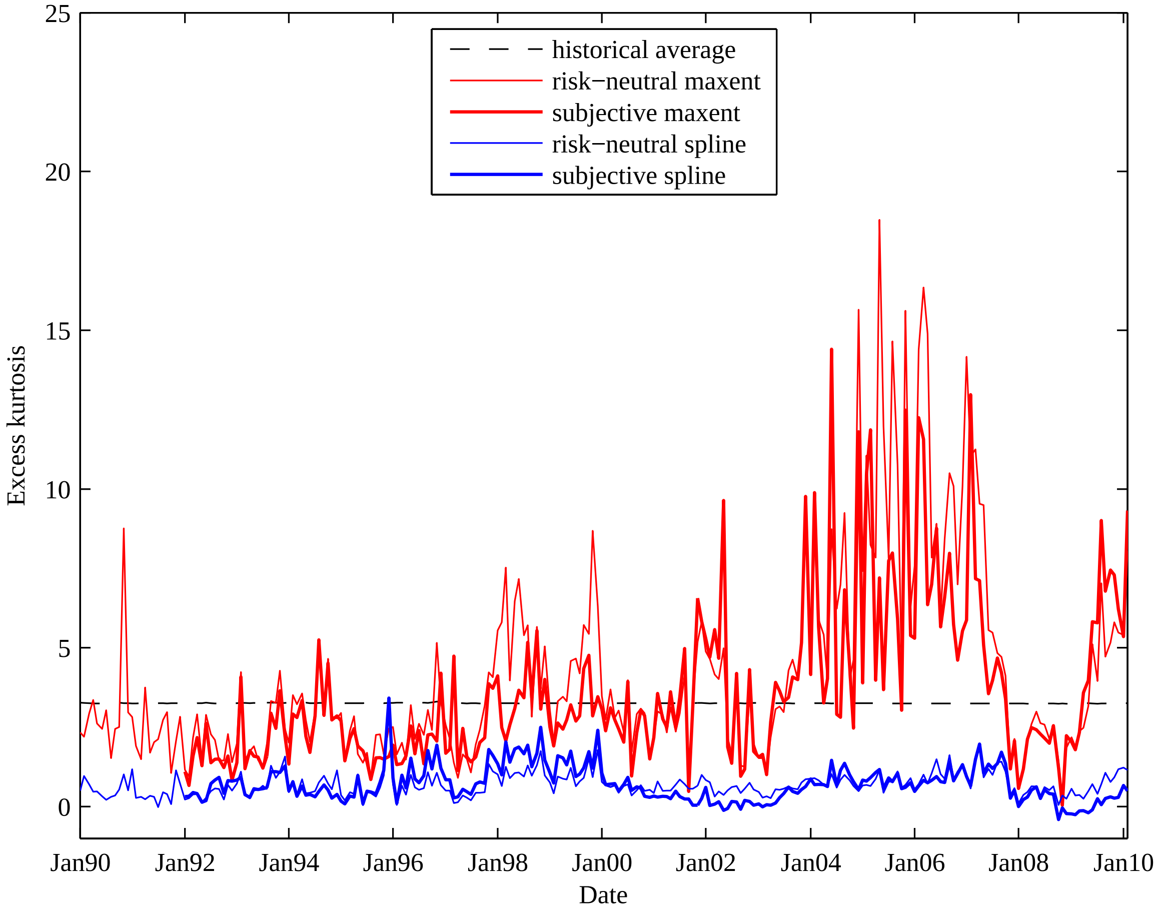


Figure 5: Risk-neutral and subjective skewness forecasts



Average historical skewness is estimated by bootstrapping monthly returns from a rolling 10-year observation window. Bootstrapping is used to mitigate the impact of outliers and to avoid estimating monthly skewness from daily returns.

Figure 6: Risk-neutral and subjective excess kurtosis forecasts



Average historical excess kurtosis is estimated by bootstrapping monthly returns from a rolling 10-year observation window. Bootstrapping is used to mitigate the impact of outliers and to avoid estimating monthly kurtosis from daily returns.

Figure 7: Cumulative return of the optimal portfolios based on GJR-GARCH forecast and on option-implied subjective densities: January 1995 to March 2000

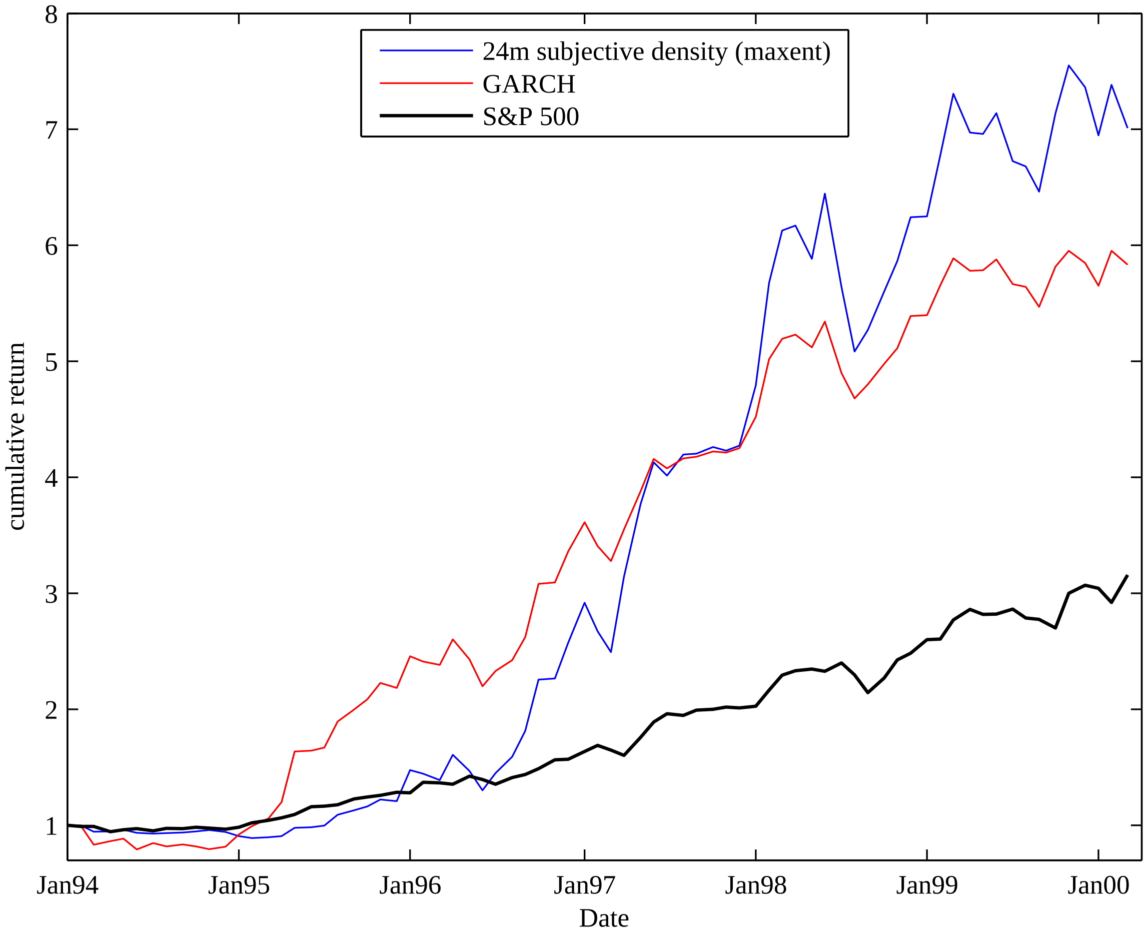


Figure 8: Cumulative return of the optimal portfolios based on GJR-GARCH forecast and on option-implied subjective densities: April 2000 to March 2010