The Equity Index Skew and

Asymmetric Normal Mixture GARCH

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Abstract

The skewness in physical distributions of equity index returns and the implied volatility skew in the risk neutral measure are subjects of extensive academic research. Much attention is now being focused on models that are able to capture time-varying conditional skewness and kurtosis. For this reason normal mixture GARCH(1,1) models have become very popular in financial econometrics. We introduce further asymmetries into this class of models by modifying the GARCH(1,1) variance processes to skewed variance processes with leverage effects. These asymmetric normal mixture GARCH models can differentiate between two different sources of asymmetry: the persistent asymmetry due to the different means in the conditional normal mixture distributions, and the dynamic asymmetry due to the skewed GARCH processes. Empirical results on five major equity indices first employ many statistical criteria to determine whether asymmetric normal mixture GARCH models can improve on asymmetric normal and Student's-t GARCH specifications. Normal, t- and normal mixture asymmetric GARCH models were also used to simulate implied volatility smiles for the S&P index, and we find that much the most realistic skews are simulated by a GARCH model with a mixture of two GJR variance components.

JEL Classification Codes: C32, G13.

Keywords: GARCH process, normal mixture, equity skew, market crash, skew persistence, leverage effect

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I Introduction

Out-of-the-money put options on an equity index are an attractive form of insurance for traders that fear a general market decline. With index option market makers in relatively short supply, the market prices of these options are often far higher than the Black-Scholes (1973) model prices based on the at-the-money volatility. Consequently the implied volatility of these options is commonly found to be higher than the implied volatility of at-the-money call and put options and out-of-the-money calls. This skew (or ‘smirk’) in equity index implied volatility has been very pronounced since the global stock market crash in 1987.¹

The equity index implied volatility skew is associated with a negatively skewed implied risk neutral returns density. Bates (1997, 2000) finds the skewness can account for around 4% of SP500 option prices and suggests that it is the risk neutral densities, rather than the physical densities, of equity indices that have changed since the crash. If the physical densities do indeed remain less skewed and leptokurtic than the risk neutral, then a possible explanation is an increase in the risk premium of the average trader since the crash of 1987.

Most option pricing models imply that both skewness and leptokurtosis in the risk neutral density should diminish substantially as maturity increases (e.g. Konikov and Madan, 2000 and others). Also, the skewness and leptokurtosis of physical returns densities should, if the central limit theorem applies, diminish as the period of returns increases. Nevertheless Carr and Wu (2003) find pronounced risk neutral skews in the SP500 index that persist into options with over one year to expiry.

Bakshi et al (2003) focus on the relationship between the higher moments of the risk neutral density and the higher moments of the physical density of index returns. They show that negative skewness in the risk neutral measure can arise even when the physical density is symmetric, provided there is excess kurtosis in the physical density and that traders have a positive risk premium. Leverage implies that the index skew should be less pronounced than individual equity skews (the effect of diversification) but Bakshi et al find that in the OEX index it can happen that the index skew is more pronounced than all individual equity skews. One ‘stylized fact’ that has emerged from this research is that, at least in the US stock market since the crash of 1987, the index skews are both too pronounced and too persistent to accord with the standard time-series analysis of the conditional densities of index returns and can only be explained by the risk premium.

The large literature on risk neutral skews is matched by an even larger literature on conditional moments in the physical measure. These are almost exclusively modelled in the generalized autoregressive conditional heteroscedasticity framework introduced by Engle (1982) and Bollerslev (1986). The observed non-normalities in both conditional and unconditional returns is higher than can be predicted by normal GARCH(1,1) models. Hence Bollerslev (1987) introduced the Student’s t-GARCH(1,1) model and Fernandez and Steel (1998) extended this to the skewed t-distribution. These t-GARCH models have no time-variation in the conditional higher moments so the t-GARCH implied equity skew may only change with maturity if there is a time varying risk premium. However many studies (e.g. Christoffersen, Heston and Jacobs, 2004; Bates, 1991 and others) emphasize the importance of time-variability in the physical conditional skewness.

More recent research by Haas et al (2004) and Alexander and Lazar (2004a) on GARCH models with mixture of normal GARCH(1,1) variance processes has shown that these models provide a better fit to physical conditional densities than many other types of GARCH models. The normal components in the mixture distribution can be interpreted as symbolising different market circumstances or groups of different investment behaviour. Since these ‘normal mixture GARCH’ models do have time-varying conditional higher moments, they could be capable of replicating volatility smiles even without a risk premium. Having said this, Alexander and Lazar (2004b) show that the continuous limit of normal mixture GARCH(1,1) models is a stochastic volatility model; not in the traditional sense of a variance diffusion, but a model with Bayesian uncertainty over the possible variance components. Since this uncertainty cannot be hedged, a risk premium must apply in the risk neutral density and the time variation in this risk premium enhances both the skew and the term structure in implied volatilities.

Alexander and Lazar (2004a) show that if the model has more than two variance components severe biases in parameter estimates are likely to result and consequently the estimated conditional skewness and excess kurtosis can be unaccountably unstable over time. At least for modelling major exchange rate time-series, they find that the mixture of two GARCH(1,1) components models outperform both symmetric and asymmetric t-GARCH models and normal mixture GARCH(1,1) models with more than two components. However, whilst the mixture of two normal variance component specifications may fit exchange rate data well, these models are not sufficiently flexible to apply to equity indices. This is because there is only one source of skewness in the physical returns densities, i.e. that arising from the different means in the components of the normal mixture conditional density. The model ignores the ‘leverage effect’ in equities that has been documented by Black (1976) and many others.3

3 Unless it is explicitly modelled, as for instance in Hansen (1994) and Harvey and Siddique (1999).
3 A fall in the equity price makes the firm more highly leveraged, and volatility increases because the firm’s future becomes more uncertain. However a commensurate rise in price will not have the symmetric effect on volatility.
Within the discrete time-varying volatility models of equity returns there is a vast literature on asymmetric GARCH models (see Engle and Ng, 1993; Glosten et al, 1993; Nelson, 1991 and many others). These models again capture only one source of skewness, i.e. the leverage effect. Clearly additional structure is required to capture the empirical observations of Bakshi et al (2003) and others about the nature of skewness in the risk neutral equity index skew. Several papers (e.g. Bekaert and Wu, 2000; Wu, 2001) study the correlation between returns and volatility. As well as emphasizing the importance of time-variability in the risk premiums, these authors show that it is the leverage effect rather than volatility feedback that determines the negative correlation between returns and volatility. Hence an asymmetric GARCH variance process should be more powerful than the GARCH-in-Mean process for explaining skew effects in the physical measure. In a recent study, Christoffersen and Jacobs (2004b) compare some GARCH models for option valuation. They also conclude that a simple asymmetric GARCH, or indeed any GARCH model that captures the leverage effect, performs best. Still, the models they considered (which did not include the normal mixture GARCH) were not able to capture either the full extent of the skewness or excess kurtosis in the data.

In this paper we first extend the normal mixture GARCH(1,1) model to introduce and differentiate two distinct sources of skewness, one ‘dynamic’ and the other ‘persistent’. We shall apply this model to historical data on five major international stock market indices, showing that it provides the best fit of the fifteen different GARCH models considered (three symmetric and twelve asymmetric GARCH models). By recovering trader’s beliefs about the likelihood of a stock market crash, and the risks and returns that may be experienced during a crash, implementing an asymmetric normal mixture GARCH allows one to draw new insights about the physical density of stock market returns. We use this (and other) GARCH models to analyse the determinants of the index skew in these stock markets. Even without a risk premium, the volatility skew implied by the asymmetric normal mixture GARCH models exhibits a pronounced skew that persists, though diminishing, for long-dated options. By contrast none of the other models can explain the observed characteristics of risk neutral skews in stock index markets without a risk premium.

The paper is structured as follows: Section II defines the general asymmetric normal mixture GARCH model and investigates the properties (such as conditional and unconditional moments) of some specific variants. Section III describes the equity index data for five major equity markets (France, Germany, UK, Japan and US) and the estimation methodology. Section IV reports the estimation results for asymmetric and symmetric normal mixture GARCH models with two variance components and for several alternative models including symmetric and skewed t-GARCH with both symmetric and asymmetric variance processes. We apply several model selection criteria to identify the best model(s). Section V examines the parameter estimates from the normal mixture GARCH models and makes inferences on the likelihood of, and behaviour during, usual market circumstances and equity
II The Asymmetric Normal Mixture GARCH Model

The specification of the model has \( K + 1 \) equations. For simplicity the conditional mean equation is written \( y_t = e_t \). It contains no explanatory variables as these can be estimated separately. The error term \( e_t \) is assumed to have a conditional normal mixture density with zero mean, which is a probability weighted average of \( K \) normal density functions with different means and variances. We write:

\[
e_t |_{t-1} \sim N \left( \left. \sum_{i=1}^{K} p_i \mu_i, \sum_{i=1}^{K} p_i \sigma_i^2 \right|_{t-1} \right), \quad \sum_{i=1}^{K} p_i = 1, \quad \sum_{i=1}^{K} p_i \mu_i = 0
\]

That is, the conditional density of the error term is

\[
\varphi(e_t) = \sum_{i=1}^{K} p_i f_i(e_t)
\]

where \( f_i \) represent normal density functions with different constant means \( \mu_i \) and different time-varying variances \( \sigma_i^2 \) for \( i = 1, \ldots, K \).

The conditional variance behaviour is described by \( K \) variance components – and these characterize, according to one interpretation, different market circumstances. These variances can follow any GARCH process but for the purpose of this paper we assume there are three possibilities. In addition to the symmetric GARCH(1,1) extensively studied in Alexander and Lazar (2004a) we consider two types of asymmetric processes:

(i) **NM-GARCH**:

\[
s_i^2 = ?_i + a_i e_{t-1}^2 + \beta_i \sigma_{t-1}^2 \quad \text{for} \quad i = 1, \ldots, K
\]

(ii) **NM-AGARCH**

\[
s_i^2 = ?_i + a_i (e_{t-1} - ?_i)^2 + \beta_i \sigma_{t-1}^2 \quad \text{for} \quad i = 1, \ldots, K
\]

(iii) **NM-GJR-GARCH**

\[
s_i^2 = ?_i + a_i e_{t-1}^2 + ?_d e_{t-1} d_{t-1} + \beta_i \sigma_{t-1}^2 \quad \text{for} \quad i = 1, \ldots, K; \quad \text{where} \quad d_{t-1} = 1 \text{ if } e_t < 0, \text{ and } 0 \text{ otherwise}
\]

In all cases the overall conditional variance is

\[
S_i^2 = \sum_{i=1}^{K} p_i s_i^2 + \sum_{i=1}^{K} p_i \mu_i^2
\]

For \( K > 1 \), the existence of second, third and fourth moments are assured by imposing less stringent conditions than in the single component (\( K = 1 \)) models. For instance, Alexander and Lazar (2004a) show that we no longer require \( a_i + \beta_i < 1 \). Indeed, Haas et al (2004) have sometimes found \( a > 1 \) on
the second and higher variance components. This way, we only require the following set of conditions for the non-negativity of variance and the finiteness of third moment.\footnote{There is no straightforward parameter constraint for existence of fourth moment. We simply require then that \(0 < E(\phi^4) < \infty\).} For \(i = 1, \ldots, K\) we must have:

\[0 < p_i < 1, \quad i = 1, \ldots, K - 1, \quad \sum_{i=1}^{K-1} p_i < 1, \quad 0 < a_i, \quad 0 \leq \beta_i < 1\]

and in the NM-GARCH model (i):

\[m = \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} \frac{p_i \beta_i^2}{(1 - \beta_i)} > 0, \quad n = \sum_{i=1}^{K} \frac{p_i (1 - a_i - \beta_i)}{(1 - \beta_i)} > 0 \text{ and } \beta_1 + a_1 \frac{m}{n} > 0\]

Also, in the NM-AGARCH model (ii) we require that:

\[m = \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} \frac{p_i (\beta_i + a_i \beta_i^2)}{(1 - \beta_i)} > 0, \quad n = \sum_{i=1}^{K} \frac{p_i (1 - a_i - \beta_i)}{(1 - \beta_i)} > 0 \text{ and } \beta_1 + a_1 \left(1 + \frac{m}{n} \right) \beta_1 > 0,\]

or, in the NM-GJRGARCH model (iii):

\[m = \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} \frac{p_i \beta_i^2}{(1 - \beta_i)} > 0, \quad n = \sum_{i=1}^{K} \frac{p_i (1 - a_i - 0.5 \beta_i)}{(1 - \beta_i)} > 0 \text{ and } \beta_1 + (a_1 + 0.5 \beta_1) \frac{m}{n} > 0\]

There are two distinct sources of asymmetry in the model:

- **Persistent Asymmetry**: This arises – in all three models – when the conditional returns density is a mixture of normal density components having different means. Appendix 1 shows that even the unconditional density will have non-zero skewness, and that this increases with the differentiation of the component means. For instance, when \(K = 2\) there is negative skewness in the overall conditional returns density when the component with the higher variance has a negative mean and positive skewness in the overall conditional returns density when the component with the higher variance has a positive mean.

- **Dynamic Asymmetry**: This is due to the \(\beta_i\) parameters in the component variance processes of models (ii) and (iii), which capture 'short-term' asymmetries due to the leverage effect. If \(\beta_i\) is positive the conditional variance in this component is higher following a negative unexpected return at time \(t - 1\) than following a positive unexpected return. In equity markets, where ‘bad news’ normally corresponds to a negative unexpected return, we expect positive \(\beta_i\). On the other hand, negative leverage coefficients may be estimated from commodity returns.\footnote{Note that the \(i\)th component of the conditional variance depends on the dispersion of the unexpected return, not around its mean \(\mu_i\) in the individual density, but around the overall mean 0. The dispersion around 0 is always greater. Hence this third effect induces more skewness in each component conditional return density than in the overall conditional return density.}

Taken together, these sources of skewness in the physical conditional returns density offer a sufficiently rich structure for capturing the behaviour of equity index skews. The unconditional moments of the two asymmetric normal mixture GARCH models (ii) and (iii) are stated in Appendix
1. Note that the unconditional skewness and excess kurtosis are non-zero, so neither the skewness nor the excess kurtosis will converge to zero: the central limit theorem does not apply. Of particular interest is the fact that the unconditional skewness in each component is zero, so the unconditional skewness in the overall index returns density stems primarily from the ‘persistent’ asymmetry, i.e. from the different means in the components of the normal mixture conditional density.

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6 Since we have conditional normality for each component, the conditional skewness for each component is zero, leading to zero unconditional skewness.
III Data and Parameter Estimation

Our results will be based on the daily closing prices of five equity market indices: CAC40, DAX30, FTSE100, Nikkei225 and S&P 500 from January 1991 to May 2003. The index prices are shown in Figures 1 - 5 and the following summarises the general characteristics of the daily returns:7

<table>
<thead>
<tr>
<th></th>
<th>CAC</th>
<th>DAX</th>
<th>FTSE</th>
<th>Nikkei</th>
<th>S&amp;P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility</td>
<td>23.4%</td>
<td>24.91%</td>
<td>18.07%</td>
<td>24.36%</td>
<td>17.45%</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.0472</td>
<td>-0.1485***</td>
<td>-0.0506</td>
<td>0.1417***</td>
<td>-0.0216</td>
</tr>
<tr>
<td>Excess kurtosis</td>
<td>2.739***</td>
<td>3.964***</td>
<td>2.657***</td>
<td>2.427***</td>
<td>3.378***</td>
</tr>
</tbody>
</table>

The skewness is negative except for the Nikkei, where it is very significantly positive. The skewness is also highly significant (and negative) in the DAX. Moderate excess kurtosis is evident, more so in the DAX and the S&P, and the FTSE and S&P indices have been less volatile than the others during this period.

For each index, we estimate the conditional variance parameters separately on the residuals $e_t$ from AR($p$) conditional means equations. All indices had significant positive autocorrelation at this daily frequency and, using information criteria we identified up to $p = 4$ lags.8 Then, maximizing the likelihood, or equivalently, maximizing

$$L (\theta | e) = \sum_{t=1}^{T} \ln [\mathcal{N}(e_t)]$$

gives the optimal parameter values, given the data. One major problem in any type of optimisation is the search for appropriate starting values, to ensure that the optimisation process leads to the global optimum, instead of a local one. To overcome this problem, as suggested by Doornik (2000), an initial grid search is performed. However, the difficulty of optimisation increases with the number of parameters, thus with the number of components in the mixture.

The updating formula has the following form, where $\mathbf{g}$ is the gradient vector, $\mathbf{H}$ the Hessian matrix and $s$ represents the step-length:

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7 For an (annualised) return $X$ the first four moments of its distributions are the mean $\mu = \mathbb{E} (X)$, variance $\sigma^2 = \mathbb{E} [(X - \mu)^2]$, skewness, $t = \mathbb{E} [(X - \mu)^3] / \sigma^3$ and excess kurtosis, $k = \mathbb{E} [(X - \mu)^4] / \sigma^4 - 3$. The standard error (s.e.) of the sample estimates of these parameters are as follows: s.e. sample mean $= \sigma / \sqrt{T}$, s.e. sample variance $= \sqrt{2} \sigma^2 / T$, s.e. of the sample skewness $= \sqrt{6 / T}$, s.e. of the sample excess kurtosis $= \sqrt{24 / T}$, where $T$ represents the total number of observations. In the table *** represent results significantly different from zero at the 0.1% level.
8 We fitted the following AR($p$) models:

<table>
<thead>
<tr>
<th>Index</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC:</td>
<td>$r_t = 0.0039 + 0.0405 r_{t-1} - 0.0417 r_{t-2} - 0.0494 r_{t-3} + 0.0095 r_{t-4} + e$</td>
</tr>
<tr>
<td>DAX:</td>
<td>$r_t = 0.0044 + 0.0066 r_{t-1} - 0.0381 r_{t-2} + e$</td>
</tr>
<tr>
<td>FTSE:</td>
<td>$r_t = 0.0036 + 0.0218 r_{t-1} - 0.0506 r_{t-2} - 0.0493 r_{t-3} + e$</td>
</tr>
<tr>
<td>NIKKEI:</td>
<td>$r_t = -0.0056 - 0.0269 r_{t-1} - 0.0485 r_{t-2} + e$</td>
</tr>
<tr>
<td>SP:</td>
<td>$r_t = 0.0062 + 0.0133 r_{t-1} + e$</td>
</tr>
</tbody>
</table>
\[ y_{t+1} = \mu_{t+1} + \varepsilon_{t+1} \]

To compute the Hessian matrix and the gradient vector, we can use either analytic or numerical first and second order derivatives of \( L(\theta | e) \) with respect to \( \theta \) - see Appendix 2.

IV Empirical Results

We fitted three symmetric and twelve asymmetric GARCH models to the equity index data:

1. Symmetric normal GARCH(1,1)
2. Normal AGARCH(1,1)
3. Normal GJR(1,1)
4. Symmetric GARCH(1,1) with Student’s t distributed errors
5. AGARCH(1,1) with Student’s t distributed errors
6. GJR(1,1) with Student’s t distributed errors
7. GARCH(1,1) with Skewed Student’s t distributed errors
8. AGARCH(1,1) with Skewed Student’s t distributed errors
9. GJR(1,1) with Skewed Student’s t distributed errors
10. Symmetric NM(2)-GARCH(1,1) with zero means in the mixture
11. NM(2)-AGARCH(1,1) with zero means in the mixture
12. NM(2)-GJR(1,1) with zero means in the mixture
13. NM(2)-GARCH(1,1)
14. NM(2)-AGARCH(1,1)
15. NM(2)-GJR(1,1)

The estimations are reported in Tables 1 – 5. The upper figure in each cell reports the parameter estimate and the lower figure is the t-ratio. Note that in these tables the first row reports the degrees of freedom for the t-GARCH models (4) – (9) and the highest weight of the two components in the NM(2)-GARCH models. Similarly, the third row reports the skewness parameter for models (7) – (9) and the mean of the first normal density for the normal mixture GARCH models.

Model Selection

The four model selection criteria used are

(a) The maximum likelihood: To account for parsimony the Akaike Information Criterion (AIC) and Schwartz’s Bayesian Information Criterion (BIC) were also examined (though not reported in the Tables).

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9 The results were generated using C++ and Ox version 3.30 (Doornik, 2002) and the G@rch package version 3.0 (Laurent, S. and Peters, J.-P., 2002)
10 In Tables 1 – 5 the parameters are estimated by MLE. Numbers in parenthesis represent t-ratios with * and ** signifying significance at the 5% and 1% level respectively.
(b) The Newey (1985) moment specification tests: Following Newey (1985), we test for normality in the standardized residuals, checking the first four moments and for zero autocorrelations in the powers, using a Wald test. There are a total of 20 conditions including a joint test on all conditions. Test statistics for the moment tests have a $\chi^2(1)$ distribution and for the cumulative test have a $\chi^2(20)$ distribution. The Tables report the number of tests (out of 20) that are rejected at 1%.

(c) The unconditional density test: The density test is on the histogram fit between the model simulated data and the original data. This is one of the most difficult tests for GARCH models to pass as it tests for the unconditional distributional fit. The model returns are simulated and their histogram is estimated using a nonparametric kernel approach. Several alternatives are available for the kernel, our chosen function being that of Epanechnikov (1969). Then the model selection criterion is based on the modified Kolmogorov-Smirnov (KS) statistic (Kolmogoroff, 1933, Smirnov, 1939, Massey, 1951 and Khamis, 2000).

(d) The Autocorrelation Function (ACF) test: By contrast to (iii) this test captures the dynamic properties of the model squared returns – namely, the empirical autocorrelations of the squared returns. Appendix 3 states the theoretical autocorrelation functions of the different models and we apply the Mean Squared Error (MSE) criterion to assess the fit with the empirical autocorrelations.

The results of these specification tests are shown in the last four rows of Tables 1 – 5. These are discussed in turn:

(a) Likelihood: All series favour the NM(2)-GJRGARCH model with non-zero means in the components, except the Nikkei for which the NM(2)-AGARCH is preferable. This is not just because there are more parameters in these models, as the AIC supports the likelihood results. Note that the BIC prefers the t-GARCH models (4) – (6) for some series.

(b) Moment specification tests: These tests show that the most basic models, i.e. (1) – (3) do not capture the higher moments. But beyond this observation, the moment tests do not distinguish well between the models. We find that either most of the models pass all tests (as in the case of the FTSE index), or most have several rejections (as in the DAX). Overall, we can say that the t-GARCH models (4) – (9) produce marginally better results on these moment tests.

(c) Unconditional density: This shows a clear preference for the NM(2)-GJRGARCH with all three sources of asymmetry (i.e. with different component means), except for the CAC (which prefers zero means).

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11 We simulate returns, based on the estimated parameters, and to ensure that the simulated density is not affected by small sample size we use 50,000 replications. Also, to avoid any influence of the starting values, each simulation has 1000 steps ahead in time but we only use the last simulated return.
(d) A CF test: This test also favours the asymmetric NM(2) models. First note that whilst one (or more) of the t-GARCH models may do a reasonable job to capture the dynamic properties of the squared residuals, these models perform badly for at least one index. The best t-GARCH model according to this criterion is model (5), AGARCH(1,1) with Student’s t distributed errors. Nevertheless this still has MSE of 3.1054 in the DAX, 1.2539 in the Nikkei and 1.8548 in the S&P 500. By contrast, all the NM(2) models (10) – (15) perform very well according to this criteria.

In summary,

- The two most important tests (c) and (d) indicate a clear superiority of the NM(2)-GARCH models over t-GARCH models and the simple symmetric and asymmetric GARCH models, although slightly different specifications do better for different series.
- The ‘persistent’ source of asymmetry appears to be important in all indices, except the CAC. That is, the non-zero means models (14) and (15) are generally preferred.
- The ‘dynamic’ asymmetries (i.e. those due to leverage) are also very important – that is, both types of asymmetric components - AGARCH and GJR-GARCH - greatly improve the fit.
- The NM(2)-GARCH models perform well according to criteria (a) and (b) as well.
- The normal GARCH(1,1) model is the worst fit by all criteria.
- The Students t-GARCH models (4) – (9) fit well according to (a) with the BIC criteria, and (albeit along with other models) they also do well on the moment tests. However, unless an AGARCH or GJR specification is used, the models yield a ridiculously high unconditional volatility.
- Interestingly, the models (7) – (9) don’t perform much better than the models (4) – (6). Thus, when the variance process is either AGARCH or GJR-GARCH, the additional asymmetry afforded by using the skewed t-distribution in place of the standard Student’s t-distribution appears to be unnecessary.

V Normal Mixtures: GARCH with ‘Normal’ and ‘Crash’ Components

We now focus on the parameter estimates for the asymmetric normal mixture GARCH models (11), (12), (14) and (15). From the results in Alexander and Lazar (2004b) we know that the continuous limit of the normal mixture GARCH model is a stochastic volatility normal mixture diffusion process with normal mixture transition densities and normal mixture marginal density. The mixing law corresponds, in behavioural terms, to the uncertain views about volatility that are held in a homogeneous population of traders. Since in all of the estimated normal mixture GARCH models for equity indices we have a high volatility component with a low probability and a low probability component with a high probability, it is clear that these normal mixture GARCH models are capturing
a ‘usual market circumstances component’ and a ‘crash component’ in equity markets. The probabilities associated with these components represent the trader’s beliefs, i.e. the probabilities he assigns to a normal market and to the occurrence of a crash during the forecast horizon.

One limitation of the normal mixture GARCH framework is that the state probability is assumed constant i.e. traders have no learning opportunity, so that their prior beliefs are not updated. If traders are to learn then the state probabilities would need to be time-varying, as for instance in the class of Markov Switching GARCH Models introduced by Hamilton and Susmel (1994) and Cai (1994). Nevertheless, normal mixture GARCH models already give considerably more insight to the behaviour of physical returns densities for equity indices than has previously been recovered from physical data.

In particular the first component (i.e. ‘usual’ market component) has the following features:

- A high probability: a weight of approximately 0.96 (CAC, DAX), 0.92 (FTSE), 0.95 (Nikkei) and 0.93 (S&P).
- An annualised return of 0.4% (CAC), 0.3% (DAX), 0.6% (FTSE), - 0.4% (Nikkei) and 0.6% (S&P). Note that the Japanese market is the exception, having a negative expected return under ‘usual’ market circumstances.\(^{12}\)
- A lower unconditional variance, with a volatility of approximately 21% (CAC, DAX), 16% (FTSE), 23% (Nikkei) and 15% (S&P).
- Usual GARCH parameter values.
- The leverage effect is significant, has the expected sign, and it is not too strong.

The second component (i.e. the ‘crash’ component) has the following features:

- A low probability: a weight of between 0.04 and 0.08 in the mixture
- A negative mean return (except in the Nikkei, so here the second component is actually an upward jump component). It varies considerably, much more than the mean of the ‘usual’ market component: from - 3% for the FTSE to - 8% for the CAC and the S&P.
- A high unconditional volatility, of around 45 – 50% in the CAC, DAX and Nikkei, but much lower (at around 30%) for the FTSE and S&P.
- A significantly higher constant, a lower persistence parameter, and a larger reaction parameter than in the first component.
- A very pronounced leverage effect during ‘crash’ markets in the S&P and CAC. Note that this is not apparent from the simple skewness statistics estimated on index returns.

\(^{12}\) When the means are non-zero (as in models (14) and (15)).
An important advantage of the normal mixture GARCH models is the ability to model time variation in the conditional higher moments of returns, i.e. time varying conditional skewness and kurtosis. In order to use these estimates, we need first to compare the unconditional higher moments of returns estimated by the normal mixture GARCH models with the unconditional higher moments of returns estimated using simple sample statistics:

- For all indices except the Nikkei, the negative skewness arises from having a low variance component with positive mean, and negative mean on the high variance component. The positive skewness in the Nikkei is captured by the negative mean on the low variance component (and corresponding positive mean on the high variance component).

- For the DAX, FTSE and Nikkei, the models’ unconditional skewness is commensurate with the sample statistics in Section III. However, for the S&P the normal mixture GARCH models reveal more negative skewness than direct estimation of unconditional skewness on returns data. This happens because the leverage coefficient estimate in the second component is exceptionally large in the S&P.

- When symmetric GARCH(1,1) components are used (as in models (10) and (13)), generally the models’ excess kurtosis is much too high. However the introduction of asymmetric components brings the excess kurtosis down to realistic levels, very close to those observed in the sample moments.

We deduce that the additional asymmetries introduced by AGARCH or GJRGARCH components in the normal mixture GARCH models are very important for the accurate modelling of higher moments of returns.

The NM(2)-GARCH with asymmetric components has been able to distinguish several behavioural characteristics of the five major world equity markets for a time series analysis of the returns data alone. The main features of the crash periods are as follows:

1. Crash periods in the **US market** carry a probability of around 6%. This market is characterized by very large downward jumps, on average around -8% in annual terms. The average volatility during crashes is around 30%. Also, at these times the leverage effect is very pronounced.

2. Crashes in the **UK market** occur with a probability of around 8%, but these are accompanied by smaller jumps of approximately -2%. The average volatility at crash times is similar to the one in the US but the leverage effect is less significant.

3. In the **French and German markets** crashes have an occurrence rate of 4% and the jumps during such periods average at -8%. But the market is more volatile during market crashes.

---

13 These coefficients are also large in the CAC. However it was the models without leverage that provided a better fit to the CAC.
than in the other countries, with a volatility of around 45%. The leverage effect is also significant, but slightly less than in the US.

4. The **Japanese market** behaves very differently from the other four markets as the market has been in decline for many years. There are no crash periods, in fact the unusual state is the one in which there is an upward jump in the index! This state does not happen too often (with a probability of only 5%) and then the average volatility is more than 50%, the expected return is positive and there is a more pronounced negative skew.

VI **Equity Index Implied Volatility Skews**

A 'stylized fact' that has emerged from recent literature (reviewed in the introduction) is that index skews are both too pronounced and too persistent to accord with the standard time-series analysis of the conditional densities of index returns. But the asymmetric normal mixture GARCH process is not a standard time-series model. It has time-varying conditional skewness and kurtosis, so the volatility skew will not be the same for all maturities. It is also able to distinguish between two sources of asymmetry in physical returns distributions - a dynamic leverage effect and a more persistent asymmetry in the skew. Furthermore, it recovers traders' beliefs about the likelihood of a market crash, and the returns and volatility behaviour during tranquil and crash periods.

A natural question to ask, therefore, is how these properties are reflected in the equity skews implied by these models. In this section we compare the implied volatility skews generated by asymmetric normal mixture GARCH models with those implied by other GARCH models, in the physical measure. In this section we show that the asymmetric normal mixture GARCH model generates realistic skews, even in the absence of a risk premium. We subsequently differentiate between two asymmetric effects: a persistence in the skew that is captured by the difference in means of the variance components, and a leverage effect that is captured by the asymmetry in each variance component.

We shall use the parameter estimates of the S&P index returns given in Table 5 for: the normal, t- and asymmetric t-GARCH(1,1) models; the GJR parameterizations for the standard, t- and asymmetric t-GARCH models; and the zero-mean NM(2)-GARCH with symmetric and skewed (based on the GJR model) components.\(^\text{14}\) We now simulate volatility skews using each of these models and compare their characteristics.

Since our results are based on daily returns, we also simulate daily returns. Starting with \(S_0 = 100\) and using \(r = 0.03\), we simulate the dynamics of the index value as:

\(^{14}\) Simulated skews for the other equity indices are available from the authors on request.
A European call option price of strike $K$ and maturity $T$ is computed as:

$$c = \exp (-rt) \mathbb{E} \left( \max (0, S_T - K) \right)$$

Repeating this procedure 50,000 times and computing their average gives our estimate of the option price. Then, applying the inverse Black-Scholes formula gives the simulated implied volatility at $(K, T)$.

Figure 1 presents the smiles based on the nine models mentioned above. For comparison, we have tried to use the same vertical scale from 0 – 25% volatility for each smile. Since it has recently been shown that the continuous limit of the normal GARCH(1,1) model has a time varying deterministic variance process, there should be no risk premium in this model. Notice that the normal GARCH(1,1) skew is almost completely flat, there being nothing in the model to capture asymmetry or term structure, except a mean reversion in the deterministic variance process.

The GJR skews in figures (d) – (f) are more realistic, with substantially higher volatility for ITM calls than OTM calls. Also, the additional asymmetry afforded by using the skewed $t$-distribution in place of the standard Student’s $t$-distribution appears to be unnecessary, as there is very little difference between figure 1(e) and 1(f). Interestingly, we reached exactly the same conclusion based on our statistical tests. But again, there is no uncertainty in the model and hence we find very little term structure in the skew.

As expected, the NM(2)-GJR model produces much the most realistic skew: not only is the skew pronounced (and less linear than in the single component GJR model), there is much more variation of volatility over time. Comparing figure 1(g) with 1(h), there is a small increase in the skew’s persistence when non-zero means are admitted in figure 1(h). But it is small. On the other hand, comparing figure 1(i) with the best of the single component GJR parameterizations (arguably figure 1(f)) the additional component in figure 1(i) allows for a richer structure in the skew with a noticeable term structure, exactly as we had hoped.

VII Summary and Conclusions

This paper has introduced a new type of asymmetry into what is, arguably, the most successful and tractable GARCH model in common use, i.e. the normal mixture GARCH(1,1) model. Even without

15 However, for the two $t$-GARCH skews (figures 1(b) and 1(c)) this was impossible. Recall from Table 5 that the long term volatility estimates from these models were improbably high (at 41.21% for the $t$-GARCH and 32.43% for the asymmetric $t$-GARCH). No surprise then that their volatility skews are completely unrealistic.

asymmetry in the variance components, the symmetric normal GARCH(1,1) already has time-varying conditional skewness and kurtosis. So is it necessary to add a further source of asymmetry, using for instance GJR and AGARCH component variance processes?

The answer to this question is undoubtedly yes. Both the statistical criteria and the simulations of the index skew justify the addition of this second type of asymmetry. The GJR (or AGARCH) components capture a leverage effect whilst the different component means capture a more persistent skew effect. It is only with different component means that the unconditional skewness is non-zero in this case. But our results show that the dynamic asymmetry of the GJR (or AGARCH) components appears to be much the most important effect in equity markets. Our battery of statistical tests shows that the addition of dynamic asymmetry is very highly significant and dramatically improves the fit of the normal mixture model. However, the skew persistence that is captured by the use of two components with different means is only marginally significant.

To summarize our empirical findings, we have demonstrated the clear superiority of normal mixture GARCH models over any single component GARCH models including the GJR parameterization and the skewed and standard Student’s t-GARCH models. Our results are also supported by a very powerful behavioural interpretation for the theoretical models, where traders’ beliefs about the likelihood of a crash, and the returns and volatility in tranquil times and during the crash period, can be recovered from the physical data. Over the data period considered (January 1991 to May 2003) the perceived likelihood of a crash was least in the Japanese, French and German markets (about 4%); next comes the US with the crash likelihood of about 6% and finally the UK, where traders’ associate a probability of 8% to the crash scenario. We have found that the crash market in the US is much stronger than it is in the UK, in that the index jumps down further. Nevertheless, the UK and US markets have the lowest crash volatilities, of 30% compared with 45-50% in the Japanese, French and German markets. There is a very pronounced leverage effect during crash markets in the US and France and this, since traders believe that the market could crash (albeit with a low probability) the leverage effect dominates the long term persistence features in the equity skew.

Our future research will first extend the limiting results of Alexander and Lazar (2004b) to the case of asymmetric GARCH variance components and thus derive the functional form of the time varying risk premium. Comparison of the risk neutral skews generated by the NM(2)-GJR model would be of particular interest. Then the discrete and continuous versions of the NM(2)-GJR model could be calibrated to physical and options data simultaneously, following the work of Chernov and Ghysels (2000) and others. Thereafter, armed with tractable lognormal mixture transition and marginal price densities, the model should provide a useful tool for pricing and hedging path dependent options.
References


Appendix 1: Moments of the Asymmetric Normal Mixture GARCH Models

We use the following notations: \( x = E (e^2_t) = E (s_t^2) \) and \( y_i = E (s_t^2) \) for \( i = 1, \ldots, K \).

Taking expectations of (1) and (2) gives for NM-AGARCH

\[
x = E (e^2_t) = E (s_t^2) = \frac{\sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} p_i \left( \gamma_i + \alpha_i \beta_i \right)}{1 - \sum_{i=1}^{K} \frac{p_i \alpha_i}{\left(1 - \beta_i\right)}}
\]

\[
y_i = (1 - \beta_i)^{-1} \left( \gamma_i + \alpha_i (x + \gamma_i^2) \right) \quad i = 1, \ldots, K
\]

Taking expectations of (2) and denoting \( a_i + 0.5 \beta_i \) by \( d_i \) gives for NM-GJRGARCH

\[
x = E (e^2_t) = E (s_t^2) = \frac{\sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} p_i d_i^2}{1 - \sum_{i=1}^{K} \frac{p_i d_i}{\left(1 - \beta_i\right)}}
\]

\[
y_i = (1 - \beta_i)^{-1} \left( \gamma_i + d_i x \right) \quad i = 1, \ldots, K
\]

assuming (approximately) that \( d_i^- \) and \( e_i^2 \) are independent.

Taking expectations of (3) gives:

\[
x = \sum_{i=1}^{K} p_i y_i + \sum_{i=1}^{K} p_i \mu_i^2
\]

The third moment is:

\[
h = E (e^3_t) = \sum_{i=1}^{K} p_i E_i (e^3_t) = \sum_{i=1}^{K} p_i (3 y_i + \mu_i^3)
\]

and the skewness can be expressed as \( s = \frac{h}{x^{3/2}} \).

The excess kurtosis in both models is:

\[
? = \frac{E (e^4_t)}{E (e^2_t)^2} - 3 = \frac{z}{x^2} - 3
\]

where

\[
z = E (e^4_t) = \frac{3p^T B^{-1} f - s}{1 - 3p^T B^{-1} g}
\]

The fourth moment uses the following notation:
\[
\mathbf{p} = (p_1, \ldots, p_k)'
\]

\[
s = \sum_{i=1}^{K} p_i \left(6\mu_i^2 y_i^2 + \mu_i^4\right)
\]

\[
\mathbf{B} = \begin{bmatrix}
1 - \beta_1^2 - 2d_1 \beta_1 e_{11} & -2d_1 \beta_1 e_{12} & \cdots & -2d_1 \beta_1 e_{1K} \\
-2d_1 \beta_2 e_{21} & 1 - \beta_2^2 - 2d_2 \beta_2 e_{22} & \cdots & -2d_2 \beta_2 e_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
-2d_k \beta_k e_{k1} & -2d_k \beta_k e_{k2} & \cdots & 1 - \beta_k^2 - 2d_k \beta_k e_{kK}
\end{bmatrix}
\]

where \(d_i = \begin{cases} a_i & \text{in (1)} \\ a_i + 0.5?_i & \text{in (2)} \end{cases}\), and \(e_{ij} = a_{ij} p_j\)

\[
\mathbf{g} = \begin{bmatrix}
?_1 + 2d_1 \beta_1 d_1 \\
?_2 + 2d_2 \beta_2 d_2 \\
\vdots \\
?_k + 2d_k \beta_k d_k
\end{bmatrix}
\]

\[
\mathbf{f} = \begin{bmatrix}
w_1 + 2d_1 \beta_1 c_1 \\
\vdots \\
w_K + 2d_k \beta_k c_k
\end{bmatrix}
\]

Furthermore:

\[
w_i = \begin{cases} ?_i^2 + x \left(2?_i a_i + 6a_i^2 ?_i^2\right) + a_i^2 \left(?_i^4 - 4?_i h\right) + 2?_i a_i ?_i^2 & \text{in (1)} \\
?_i^2 + 2?_i d_i x + 2?_i \beta_i y_1 & \text{in (2)} \end{cases}
\]

\[
c_i = \sum_{j=1}^{K} a_{ij} \left[\sum_{k=1}^{K} \frac{p_k \Gamma_{jk}}{1 - \beta_j \beta_k}\right] + y_i q \quad \text{with} \quad q = \sum_{k=1}^{K} p_k \mu_k^2
\]

\[
r_{ik} = ?_i ?_k + x \left[?_i a_k + ?_k a_i + a_i a_k \left(?_i^2 + 4?_i ?_k + ?_k^2\right)\right] - 2a_i a_k h(?_i + ?_k)
\]

\[
r_{ik} = ?_i ?_k + x \left[?_i d_k + ?_k d_i \beta_1 \beta_k \right] + \beta_i y_i \left[?_k + a_k ?_i^2\right] + \beta_k y_k \left[?_i + a_i ?_k^2\right] + \beta_i a_k \beta_k ?_i^2 + a_i a_k ?_k^2 ?_i^2
\]

The full derivation of these results is available from the authors on request.
Appendix 2: Numerical Derivatives of the Asymmetric Normal Mixture GARCH Models

The only difference from the NM(K) -GARCH model numerical derivatives (Alexander and Lazar, 2004a) is the first and second order derivatives of $s_t^2$ with respect to $\alpha_i$ and these are as follows:

\[
\frac{\partial \sigma_{it}^2}{\partial \alpha_i} = z_{it} + \beta_i \frac{\partial \sigma_{i,t-1}^2}{\partial \alpha_i}
\]
\[
\frac{\partial^2 \sigma_{it}^2}{\partial \alpha_i \partial \alpha_i} = w_{it} + \beta_i \frac{\partial^2 \sigma_{i,t-1}^2}{\partial \alpha_i \partial \alpha_i}
\]

where \( w_{it} = A_{it} + A_{i,t}^{\alpha} \)

For NM-AGARCH:

\[
z_{it} = (1, (e_{i-1} - \alpha_i)^2, -2a_i (e_{i-1} - \alpha_i), s_{i-1}^2, s_{i-1}^2)' \]

The starting values for this expression (for \( t=0 \)) are:

\[
\frac{\partial s_{i0}^2}{\partial \alpha_i} = (1, s^2 + \gamma_i^2, 2a_i \alpha_i, s_{i-1}^2, s_{i-1}^2)' \]

where \( s^2 = \frac{\sum s_i^2}{T} \)

\[
A_{it} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 2(e_{i-1} - \alpha_i) & a_i & 0 & 0 \\
\end{bmatrix}
\]

The starting values for this computation are

\[
\frac{\partial^2 s_{i0}^2}{\partial \alpha_i \partial \alpha_i} = \frac{1}{(1 - \beta_i)} \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 2\gamma_i & s^2 + \gamma_i^2 \\
0 & 2\gamma_i & 2a_i & 2a_i \gamma_i \\
1 & s^2 + \gamma_i^2 & 2a_i \gamma_i & 2s^2 \\
\end{bmatrix}
\]

For NM-GJRGARCH:

\[
z_{it} = (1, e_{i-1}^2, d_{i-1}^\alpha e_{i-1}^2, s_{i-1}^2)' \]

The starting values for this expression (for \( t = 0 \)) are:

\[
\frac{\partial s_{i0}^2}{\partial \alpha_i} = (1, s^2, 0.5s^2, s_{i-1}^2, s_{i-1}^2)' \]

where \( s^2 = \frac{\sum s_i^2}{T} \)

\[
A_{it} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and the starting values for this computation are

\[
\frac{\partial^2 s_{i0}^2}{\partial \alpha_i \partial \alpha_i} = \frac{1}{(1 - \beta_i)} \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & s^2 \\
0 & 0 & 0 & 0.5s^2 \\
1 & s^2 & 0.5s^2 & 2s^2 \\
\end{bmatrix}
\]
Appendix 3: The Autocorrelation Function of the Squared Errors in the Asymmetric Normal Mixture GARCH Models

The autocorrelations of the squared errors can be expressed as:

\[ c_k = \text{Corr}(e_t^2, e_{t-k}^2) = \frac{\text{Cov}(e_t^2, e_{t-k}^2)}{\text{Var}(e_t^2)} = \frac{E[e_t^2 e_{t-k}^2] - x_t^2}{E[e_t^2] - x_t^2} = c_k - \frac{x_t^2}{z - x_t^2}, \]

\[ c_k = E[e_t^2 e_{t-k}^2] = \sum_{i=1}^{K} l_i \mu_i^2 + \sum_{i=1}^{K} l_i E[s_i^2 e_{t-k}^2] = \sum_{i=1}^{K} l_i \mu_i^2 + \sum_{i=1}^{K} l_i p_i b_{ik} \]

For NM-AGARCH:

\[ b_{ik} = (\theta_1 + a_i \theta_1^2) x_t + a_i c_{k-1} + \beta_b b_{ik-1}, \quad k > 1 \]
\[ b_{i1} = (\theta_1 + a_i \theta_1^2) x_t + a_i c_0 + \beta_b b_{i0} + -2a_i \theta_1 \theta_1, \quad k = 1 \]

For NM-GJRGARCH:

\[ b_{ik} = \theta_x x_t + (a_i + 0.5 \theta_i 2) c_{k-1} + \beta_b b_{ik-1} \]

The starting values are: \( c_0 = z \) and \( b_{i0} = c_i + d_x z + e_i' B^{-1}(f + g_x) \).

\[ \text{Since the variance of the NM(K)-GARCH(1,1) model can be expressed as a GARCH(K,K) variance, according to Bollerslev (1986) the autocorrelations can also be written as an AR(K) process.} \]
### Table 1. Estimation results for the CAC 40 Index

<table>
<thead>
<tr>
<th>Model</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
<th>(13)</th>
<th>(14)</th>
<th>(15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_{1}(d.f. \text{ for (4)-(9)}))</td>
<td>9.7319</td>
<td>10.2452</td>
<td>10.2712</td>
<td>9.8402</td>
<td>10.3367</td>
<td>10.3913</td>
<td>0.9423</td>
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<td>0.9660</td>
<td>0.9590</td>
<td>0.9450</td>
<td>0.9270</td>
<td>0.9150</td>
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<tr>
<td>(\mu_{1}(S)\text{ for (7)-(9)})</td>
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<td>-0.0245</td>
<td>-0.0292</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0046</td>
<td>0.0039</td>
<td>0.0037</td>
<td>0.0048</td>
<td>0.0047</td>
<td>0.0115</td>
<td>0.0048</td>
<td>0.0047</td>
<td>0.0115</td>
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<tr>
<td>(a_{1})</td>
<td>0.0653</td>
<td>0.0458</td>
<td>0.0120</td>
<td>0.0583</td>
<td>0.0522</td>
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<td>0.0143</td>
<td>0.0488</td>
<td>0.0470</td>
<td>0.0116</td>
<td>0.0488</td>
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<td>0.0116</td>
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<tr>
<td>(?_{1})</td>
<td>1.2E-3</td>
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<td>9.9E-4</td>
<td>6.8E-4</td>
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<td>(?_{2})</td>
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<td>0.9347</td>
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<td>21.83%</td>
<td>21.83%</td>
<td>22.75%</td>
<td>22.27%</td>
<td>22.27%</td>
<td>22.75%</td>
<td>22.26%</td>
<td>22.24%</td>
<td>23.01%</td>
<td>23.10%</td>
<td>23.20%</td>
<td>23.29%</td>
<td>23.10%</td>
<td>23.20%</td>
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<tr>
<td>Unconditional s2</td>
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<td>43.73%</td>
<td>42.23%</td>
<td>41.22%</td>
<td>46.56%</td>
<td>41.30%</td>
<td>41.05%</td>
<td>43.73%</td>
<td>42.23%</td>
<td>41.22%</td>
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<tr>
<td>Unconditional t</td>
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<td>-0.0407</td>
<td>-0.0480</td>
<td>-0.0344</td>
<td>-0.0407</td>
<td>-0.0480</td>
<td>-0.0344</td>
<td>-0.0407</td>
<td>-0.0480</td>
<td>-0.0344</td>
<td>-0.0407</td>
<td>-0.0480</td>
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Table 2. Estimation results for the DAX 30 Index

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Fig. 1. Simulated Equity Index Skews

(a) Normal GARCH(1,1)

(b) t-GARCH(1,1)

(c) Asymmetric t-GARCH(1,1)
(g) NM(2)-GARCH(1,1) zero means

(h) NM(2)-GARCH(1,1)

(i) NM(2)-GJR