These worked examples and the accompanying Excel workbook provide a more detailed understanding of the concepts introduced in this lecture. The solution to each question is illustrated with an Excel spreadsheet: in these anything **red** is an input that can be changed, and anything **blue** is an output.

Question 1: VaR at Portfolio Level [Lecture Notes – Section 2]

Use the daily price series of the FTSE 100 index in the Excel spreadsheet 'Q1' to estimate the following on 22nd September 2006:

(a) 1% 1-day VaR

(b) 1% 10-day VaR

for a position of €1,000 per point on the FTSE 100 index. In each case use the following VaR models:

(i) Normal linear (analytic);

(ii) Monte Carlo (MC); and

(iii) Historical

The three models should be based on the entire returns sample and the MC VaR should use 1,000 simulations. State the assumptions of each model and comment on your results. For simplicity assume that the expected return on this position is the risk-free rate (so that no drift adjustment needs to be made to the P&L before calculating the VaR).

Solution

From the single historical price series in the spreadsheet we compute the daily log returns over the whole sample, and hence obtain the standard deviation of daily log returns for use in the linear VaR and MC VaR models.

(i) For the **linear VaR** we set:

$$\text{Linear VaR}_{h,\alpha,t} = Z_\alpha \sigma_{h,t} P_t$$

where:

- Z_α is the α% critical value of the standard normal distribution, e.g. $Z_\alpha = 2.33$ for $\alpha = 0.01$ and $Z_\alpha = 1.645$ for $\alpha = 0.05$;

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- P_t is the current nominal value of the position, in this case with €1,000 per point on the FTSE 100 [it is €58,022,300 on 22nd September 2006]; and
- $\sigma_{h,t}$ is the h-day standard deviation of returns, estimated on 22nd September 2006. For the 10-day VaR we use the square root of time rule, i.e. $\sigma_{h,t} = \sigma_{1,t} \times \sqrt{h}$ with $h = 10$.

(ii) For the **MC VaR** we take the standard deviation of daily returns that is estimated from the sample and hence simulate 1000 hypothetical one-day returns, using 'Normsinvrand' $\times \sigma_{1,t}$, as explained in Part II.G.7. We then apply the Excel percentile function to find the lower α percentile of the daily returns distribution. The lower percentile is multiplied by the nominal value of the portfolio and by $\sqrt{h}$ to convert the percentile into $\text{MC VaR}_{h,\alpha,t}$.

(iii) Finally, for the **historical VaR** the percentile is calculated on the actual daily (log) returns that were realised over the sample. Again, the lower percentile is multiplied by the nominal value of the portfolio and by $\sqrt{h}$ to convert into $\text{Historical VaR}_{h,\alpha,t}$.

The table below summarizes the results:

Simple Comparison of VaR Models

	Linear	**MC**	**Historical**
1% 1-day VaR:	€ 95,328	€ 93,571	€ 119,799
1% 10-day VaR:	€ 301,452	€ 295,898	€ 378,838

However, you will see a different value for MC VaR than that shown in this table. In fact, each time you press the F9 key, the MC VaR will change because a new set of random numbers are generated for the simulations. In this example only 1,000 simulations are used so the result can vary a lot each time we press F9. However, if we used 10,000 or more simulations, the MC VaR would be very close the linear VaR.

But why does the historical VaR model give so different a result from the linear VaR model? The reasons for this may be that:

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- The returns are not well represented by a normal distribution. In this case the linear (and MC) VaR estimates are inaccurate;
- We did not use enough data for the historical VaR model. With a less than 500 observations, the 1% lower tail of the returns distribution contains less than 5 data points, which may be too few to estimate the empirical tail with much accuracy and the historical VaR estimates are inaccurate.

Question 2: Systematic and Specific VaR [Lecture Notes – Sections 4a. and 5]

We invest 20m$ in a portfolio where the total volatility of returns is 25%. A linear model with two risk factors indicates that the portfolio has net betas of 0.8 and 1.2 with respect to the 2 risk factors. The factors have volatility 15% and 20% respectively and a correlation of −0.5. Assuming no discounting, calculate the 5% 1-month systematic VaR and also find the specific VaR, assuming this is uncorrelated with the systematic VaR.

Solution

The risk factor's monthly covariance matrix is

$$\mathbf{\Omega} = \begin{pmatrix} 0.00188 & -0.0013 \\ -0.0013 & 0.00333 \end{pmatrix}$$

so the portfolio variance due to the risk factors is

$$\boldsymbol{\beta}'\mathbf{\Omega}\boldsymbol{\beta} = \begin{pmatrix} 0.8 & 1.2 \end{pmatrix}\begin{pmatrix} 0.00188 & -0.0013 \\ -0.0013 & 0.00333 \end{pmatrix}\begin{pmatrix} 0.8 \\ 1.2 \end{pmatrix} = 0.0036$$

Hence the systematic VaR is 1.644853 × 0.06 × 20m$ = 1.973824m$. But the total portfolio volatility is 25%, so the total VaR is:

$$1.644853 \times 0.25 \times (1/\sqrt{12}) \times 20\text{m\$} = 2.374141 \text{ m\$}.$$

The specific VaR is calculated using Total VaR^2 = Systematic VaR^2 + Specific VaR^2 , giving a specific VaR of 1.319305m$.

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Question 3: Interest Rate Risk on a Hedged Foreign Investment [Lecture Notes – Section 4b.]

A UK investor buys £1m of Australian dollars on the spot market to invest in an Australian asset that has a known annual continuously compounded return of 10% per annum. The investment is for 9 months and the FX risk is hedged by selling AUD short based on the 9-month forward FX rate. Suppose these are 2.5 Australian dollars per pound for the spot rate and 2.45 Australian dollars per pound for the 9-month forward rate. What is the return on our investment in present value terms? If the UK zero-coupon continuously compounded 9-month spot rate is 5% what is the sensitivity to the UK zero-coupon rate?

Solution

The value of the investment at maturity is 2.5exp(0.1×0.75) = 2,694,710 AUD. Hence to completely hedge the FX risk we must lock in this value by selling 2,694,710 AUD at the 9-month forward FX rate of 2.45, providing a fixed Sterling amount of £1,099,882 in 9 months. Since the UK discount rate is 5%, we have £1,059,400 in present value terms, representing a continuously compounded return of 5.77% over the 9 months of our investment. But over the next 9 months the discount rate may change, so before we receive the cash we have an interest rate risk due to fluctuations in the discount rate. For instance, 1 month from now the 8-month discount rate is unknown, so the present value of the investment one month from now is uncertain. The current sensitivity to the discount rate is given by the PV01 of a cash flow of £1,059,400 in 9 months. Using the approximation (5.39) we have:

$$\text{PV01} \approx 108.56639 \times 0.75 \times \exp(-0.05 \times 0.75) = £79.45$$

The situation becomes more complex if we do not know the return on the foreign asset with certainty. The minimum variance FX hedge ratio will depend on the volatilities and correlation between the returns on the FX rate and the foreign asset.

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Question 4: Interest Rate VaR from FX Exposure [Lecture Notes – Section 4c.]

An investor buys 2m$ of sterling 10 days forward, when the 10-day treasury bill rate is 5% in the US and libor is 4.5% in the UK. If these interest rates have volatilities of 100bps for the US and 80bps for the UK, and a correlation of 0.9, calculate the 1% 10-day interest rate VaR.

Solution

The interest rate risk arises from the cash flows of 2m$ on the UK interest rate and –2m$ on the US interest rate. The PV01 vector is calculated in the spreadsheet using the method described in section 5.5.1 as $\mathbf{p} = (5.47, -5.46)'$ in US$. The annual covariance matrix of the interest rates, in basis points, is

$$\mathbf{\Omega} = \begin{pmatrix} 6400 & 7200 \\ 7200 & 10000 \end{pmatrix}$$

Now using the usual formula ($\mathbf{p}'\mathbf{\Omega}\mathbf{p}$) for the variance and calculating the 1% 10-day VaR in the usual way gives a grand total of 114$ for the interest rate VaR.

Question 5: Linear VaR From a Mapped Cash Flow [Lecture Notes – Section 4c.]

Consider a cash flow of 1m$ in one year and of 1.5m$ in two years time. Calculate the volatility of the discounted P&L of the cash flows, given that the one year interest rate is 4% and the 2 year interest rate is 5%, that the volatility of the one year rate is 100 basis points, the volatility of the two year rate is 75 basis points and their correlation is 0.9. Hence calculate the 5% 1-day VaR and the 1% 10-day VaR.

Solution:

We calculate the PV01 vector as $\mathbf{p} = -(92.465, 136.067)'$ using the method described in Part II.C.2]. Then we calculate the covariance matrix from the volatilities and correlation, in basis point terms, as:

$$\mathbf{\Omega}_{250} = \begin{pmatrix} 10000 & 6750 \\ 6750 & 5625 \end{pmatrix}$$

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Notice that we need to express the covariance matrix in basis point terms because the PV01 vector measures sensitivity to a basis point move in interest rates.

Now the volatility of discounted P&L is:

$$\sqrt{\begin{pmatrix} 92.465 & 136.067 \end{pmatrix} \begin{pmatrix} 10000 & 6750 \\ 6750 & 5625 \end{pmatrix} \begin{pmatrix} 92.456 \\ 136.067 \end{pmatrix}} = 18,960\$$$

To convert this into an α% *h*-day VaR figure we use the relevant standard normal critical value and the square root of time rule. Assuming 250 risk days per year, the 5% 1-day VaR corresponding to the volatility of 18,960$ is:

$$1.64485 \times 18,960 / \sqrt{250} = 1,972.40\$$$

Similarly, since the number of 10-day periods per year is 25, the 1% 10-day VaR is:

$$2.32634 \times 18,960 / \sqrt{25} = 8,821.48\$$$

Question 6: Incremental VaR [Lecture Notes – Section 5.]

Consider a cash flow map with the follow sensitivity vector:

Year:	*1*	*2*	*3*
PV01($):	*1000*	*1500*	*2000*

Suppose that interest rates are 4%, 4.5% and 5% at the 1-year, 2-year and 3-year vertices. The interest rates at maturities 1, 2 and 3 years have volatilities of 75bps, 60bps and 50bps respectively and correlations of 0.95 (1yr, 2yr), 0.9 (1yr, 3yr), 0.975 (2yr, 3yr). Suppose that a trader considers entering into a swap with the follow cash flow:

Year:	*1*	*2*	*3*
Cash Flow (m$)	*3*	*-3*	*-0.25*

What is the incremental VaR of the trade?

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Solution

For a 10 day risk horizon,

$$\mathbf{\Omega}_{10}\mathbf{p} = 10 \times \begin{pmatrix} 22.5 & 17.1 & 13.5 \\ 17.1 & 14.4 & 11.7 \\ 13.5 & 11.7 & 10 \end{pmatrix} \begin{pmatrix} 1000 \\ 1500 \\ 2000 \end{pmatrix} = \begin{pmatrix} 751500 \\ 621000 \\ 510500 \end{pmatrix}$$

Also $\sqrt{\mathbf{p}'\mathbf{\Omega}_h\mathbf{p}}$ = 52,000$ (giving a 1% 10-day VaR of $120,970).

Hence $$\nabla VaR_{0.01,10} = \frac{2.32634}{52000} \begin{pmatrix} 751500 \\ 621000 \\ 510500 \end{pmatrix} = \begin{pmatrix} 33.6201 \\ 27.7819 \\ 2.28384 \end{pmatrix}$$

Calculating the PV01 sensitivity vector of the swap's cash flows gives

$$\mathbf{p}^{*} = \begin{pmatrix} 277.3935 \\ -525.8534 \\ -61.7144 \end{pmatrix}$$

Hence the components of the incremental VaR are:

$$\mathbf{p}^{*} \otimes \nabla \text{VaR} = \begin{pmatrix} 277.3935 \\ -525.8534 \\ -61.7144 \end{pmatrix} \otimes \begin{pmatrix} 3.3620 \\ 2.7782 \\ 2.2838 \end{pmatrix} = \begin{pmatrix} 9{,}326 \\ -14{,}609 \\ -1{,}409 \end{pmatrix} \$$$

Hence the positive cash flow at 1year increases the VaR by 9,326$ but both of the negative cash flows on the swap will decrease the VaR, by 14,609$ and 1,409$ respectively. The total incremental VaR for the swap is the sum of these, i.e. –6,693$. Hence adding the swap would reduce the VaR of the portfolio.

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